Analysis of Algorithms WS 2022
Prof. Dr. P. Rossmanith
M. Gehnen, H. Lotze, D. Mock

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Exercise Sheet with solutions 11

Tutorial Exercise T11.1

Let $z \in \mathbf{C}$. How can we write $z^n + \bar{z}^n$ using only real numbers if $z = Re^{i\phi}$?

Solution

$$z^n + \bar{z}^n = R^n(e^{i\phi n} + e^{-i\phi n}) = R^n \cos(\phi n).$$

We can also write $z^n + \bar{z}^n = |z|^n \cos(n \arg z)$.

Tutorial Exercise T11.2

In this exercise we consider the following (regular) CFG G:

$$\begin{aligned} S &\to abA \mid bS \mid a \\ A &\to bA \mid aS \end{aligned}$$

1. Find a generating function for the number of words s_n in $L(G)$ that have length n.
2. What is the dominant singularity and what kind of singularity is it?
3. What is the exponential growth of s_n?
4. How precisely can you estimate s_n with just the knowledge of the dominating singularity and its nature?
5. Find a closed formula for s_n with an additive error of at most $O(0.8^n)$.

Solution

1. Since the grammar is unambiguous, the symbolic method gives us

   $$S(z) = z^2 A(z) + zS(z) + z,$$

   $$A(z) = zA(z) + zS(z).$$

   We solve the latter for $A(z)$ and obtain $A(z) = zS(z)/(1-z)$. Now we can insert it into the former. This yields

   $$S(z) = z^3 S(z)/(1-z) + zS(z) + z.$$

   We then solve for $S(z)$ and get the generating function

   $$S(z) = \frac{z}{1 - z - z^3/(1-z)} = \frac{z(1-z)}{(1-z)^2 - z^3}.$$

2. The singularities are the roots of of the denominator.

   We ask Maxima `solve((1-z)^2-z^3,z)` and get

$$
\begin{aligned}
z_1 &= -\frac{\left(9\sqrt{23}+11\sqrt{3}\right)^{\frac{2}{3}}\ \left(\sqrt{3}\,i+1\right)+5\,2^{\frac{2}{3}}\,3^{\frac{5}{6}}\,i-2^{\frac{4}{3}}\,3^{\frac{1}{6}}\ \left(9\sqrt{23}+11\sqrt{3}\right)^{\frac{1}{3}}-5\,2^{\frac{2}{3}}\,3^{\frac{1}{3}}}{2^{\frac{4}{3}}\,3^{\frac{7}{6}}\ \left(9\sqrt{23}+11\sqrt{3}\right)^{\frac{1}{3}}},\\
\bar{z}_1 &= \frac{\left(9\sqrt{23}+11\sqrt{3}\right)^{\frac{2}{3}}\ \left(\sqrt{3}\,i-1\right)+5\,2^{\frac{2}{3}}\,3^{\frac{5}{6}}\,i+2^{\frac{4}{3}}\,3^{\frac{1}{6}}\ \left(9\sqrt{23}+11\sqrt{3}\right)^{\frac{1}{3}}+5\,2^{\frac{2}{3}}\,3^{\frac{1}{3}}}{2^{\frac{4}{3}}\,3^{\frac{7}{6}}\ \left(9\sqrt{23}+11\sqrt{3}\right)^{\frac{1}{3}}},\\
z_0 &= \frac{\left(9\sqrt{23}+11\sqrt{3}\right)^{\frac{2}{3}}+2^{\frac{1}{3}}\,3^{\frac{1}{6}}\ \left(9\sqrt{23}+11\sqrt{3}\right)^{\frac{1}{3}}-5\,2^{\frac{2}{3}}\,3^{\frac{1}{3}}}{2^{\frac{1}{3}}\,3^{\frac{7}{6}}\ \left(9\sqrt{23}+11\sqrt{3}\right)^{\frac{1}{3}}}
\end{aligned}
$$

   Wolfram Alpha even gives us a nice diagram

   `http://www.wolframalpha.com/input/?i=(1-z)%5E2-z%5E3+%3D+0`

   from which we see that we have a small real and two larger complex conjugated singularities. We evaluate them numerically and see that their magnitudes are $|z_1| = 1.32471\ldots$ and $|z_0| = 0.5698402909980533\ldots$. The dominant singularity is z_0. We decomposed the denominator of $S(z)$ into three roots of degree one. The function $S(z)(z-z_0)$ is therefore analytic at z_0. This means that z_0 is a pole of first order.

   For a meromorphic generating function $B(z)$ with poles $\alpha_1, \ldots, \alpha_m$ we know that there exist polynomials $P_1(n), \ldots, P_m(n)$ such that

$$[z^n]B(z) = \sum_{j=1}^{m} P_j(n)\alpha_j^n$$

   and the degree of the polynomial $P_j(n)$ is one smaller than the order of the pole α_j. Since we have three poles of first order the polynomials are constants. We have

$$s_n = c_1 z_0^{-n} + c_2 z_1^{-n} + c_2 \bar{z}_1^{-n}.$$

   for some constants c_1 and c_2.

3. The exponential growth is $|z_0|^{-n} \approx 1.754877666246692655\ldots^n$.

4. Note that $|c_2 z_1^{-n}| = O(0.8^n)$. If we can find the factor c_1 we get a good approximation with a small additive error for s_n.

$$
\begin{aligned}
S(z) = \frac{z}{1-z-z^3/(1-z)} &= \frac{z(1-z)}{(1-z)^2-z^3} . \frac{z(1-z)}{(z-z_0)(z-z_1)(z-\bar{z}_1)}\\
&\sim \frac{1-z_0}{(1-z/z_0)(z_0-z_1)(z_0-\bar{z}_1)} \quad \text{for } z \to z_1
\end{aligned}
$$

   Hence $c_1 = (1-z_0)/((z_0-z_1)(z_0-\bar{z}_1)) \approx 0.23448675$ and we get the estimate

$$s_n = c_1 z_0^{-n} + O(0.8^n).$$

   We can use the following programm to verify the correctness numerically

```
s = range(0,1000)
a = range(0,1000)
s[0] = 0
```

```
s[1] = 1
a[0] = 0
a[1] = 0
for n in range(2, 100):
  s[n] = a[n-2]+s[n-1]
  a[n] = a[n-1]+s[n-1]
  print n, s[n], 1.0*s[n]/s[n-1], 0.23448675*1.754877666246692655**n/s[n]
```

The last line states

```
99 355268071453933228439241 1.75487766625 0.999999931817.
```

Indeed, after 100 iterations we only make a multiplicative error of 0.999999931817.

Homework Exercise H11.1

Prove that

$$[z^n](1-z)^w \sim \frac{n^{-w-1}}{\Gamma(-w)}$$

for $w \in \mathbf{C}$ without using the theorem of the lecture. (The idea of this assignment is to get a deeper insight into the theorem.)

Hint: Use Newton's formula and replace one of the implicit factorials by a gamma function. Remember that $\Gamma(n+1) = n!$.

Solution

The solution is straightforward. The only place where we loose precision is the approximation of a "falling" polynomial.

$$[z^n](1-z)^w = \binom{w}{n}(-1)^n = \binom{n-w-1}{n} = \frac{(n-w-1)!}{n!(-w-1)!} =$$

$$= \frac{1}{n^{\underline{w+1}}(-w-1)!} = \frac{1}{n^{\underline{w+1}}\Gamma(-w)} = \frac{n^{-w-1}}{\Gamma(-w)}\Big(1+O\Big(\frac{1}{n}\Big)\Big)$$

Homework Exercise H11.2

Approximate $[z^n]\frac{1}{2-e^z}$ up to an error of $O(12^{-n})$.

Solution

In the lecture, we found the first term (of the dominant singularity), namely $\frac{1}{2}(\frac{1}{\ln 2})^{n+1}$.

So we take a look at the singularity with the second highest absolute value, which is $\ln 2 \pm 2\pi i$. Both are poles of order 1. Let us see how $S(z)$ behaves asymptotically for $z \to \ln 2 \pm 2\pi i$. We have that $e^z \sim 2(1 - \ln 2 \mp 2\pi i + z)$ for $z \to \ln 2 \pm 2\pi i$ and therefore

$$\begin{aligned}\frac{1}{2-e^z} &\sim \frac{1}{2-(2-2\ln 2 \mp 4\pi i + 2z)} \\ &= \frac{1}{2}\frac{1}{\ln 2 \mp 2\pi i}\frac{1}{1-\frac{z}{\ln 2 \mp 2\pi i}} \\ &= \frac{1}{2}\frac{1}{\ln 2 \mp 2\pi i}\sum_{n=0}^{\infty}(\frac{1}{\ln 2 \mp 2\pi i})^n z^n\end{aligned}$$

With Theorem 9 we get:

$$
\begin{aligned}
[z^n]S(z) &= \frac{1}{2}\left(\left(\frac{1}{\ln 2}\right)^{n+1} + (\frac{1}{\ln 2 + 2\pi i})^{n+1} + (\frac{1}{\ln 2 - 2\pi i})^{n+1}\right) + O(r)^{-n} \\
&= \frac{1}{2}\left(\left(\frac{1}{\ln 2}\right)^{n+1} + r^{n+1}(e^{i\phi(n+1)} + e^{-i\phi(n+1)})\right) + O(r)^{-n} \\
&= \frac{1}{2}\left(\left(\frac{1}{\ln 2}\right)^{n+1} + 2r^{n+1}\cos(\phi(n+1))\right) + O(r)^{-n}
\end{aligned}
$$

with $r = 1/\sqrt{\ln^2 2 + 4\pi^2} \approx 12.58547409739904$, $\phi = \arctan(\frac{2\pi}{\ln 2})$.