Analysis of Algorithms WS 2022
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Exercise Sheet with solutions 05

Due date: next tutorial session

Tutorial Exercise T5.1
Let $x \in \mathbf{R}^+$. Is $\lceil \sqrt{x} \rceil = \left\lceil \sqrt{\lceil x \rceil} \right\rceil$?

Solution
Let $i \in \mathbf{N}$, s.t. $i^2 < x \leq (i+1)^2$. Then

$$i^2 < x \leq \lceil x \rceil \leq (i+1)^2.$$

Using monotonicity of $\sqrt{\cdot}$ we get

$$i = \sqrt{i^2} < \sqrt{x} \leq \sqrt{\lceil x \rceil} \leq \sqrt{(i+1)^2} = i+1$$

This means that both $\sqrt{x}$ and $\sqrt{\lceil x \rceil}$ are strictly greater than i and smaller than $i+1$. Since $i \in \mathbf{N}$ we can conclude that

$$\lceil \sqrt{x} \rceil = \left\lceil \sqrt{\lceil x \rceil} \right\rceil = i+1.$$

Tutorial Exercise T5.2
Solve the following recurrence relation by order reduction:

$$a_0 = 8000 \qquad a_1 = \frac{1}{2} \qquad a_{n+2} + a_{n+1} - n^2 a_n = n!$$

Solution

$$a_{n+2} + a_{n+1} - n^2 a_n = (E-n)\underbrace{(E+n)a_n}_{b_n}$$

Hence we have to solve the recurrence equation for b_n:

$$\begin{aligned} (E-n)b_n &= n! \\ \iff b_{n+1} - nb_n &= n! \end{aligned}$$

We guess the solution $b_n = n!$. We insert that into the recurrence relation for a_n an get:

$$\begin{aligned} (E+n)a_n &= b_n = n! \\ \iff a_{n+1} + na_n &= n! \end{aligned}$$

We get the following solution for a_n

$$a_n = \frac{(n-1)!}{2}$$

for $n > 0$ and $a_0 = 8000$.
To find the solution to $b_n = (n-1)b_{n-1} + (n-1)!$ one can use summation factors. We get

$$\begin{aligned} b_n &= (n-1)! + \sum_{j=1}^{n-1} (j-1)! \cdot j \cdot (j+1) \cdots (n-1) \\ &= (n-1)! + (n-1)(n-1)! = n!. \end{aligned}$$

Solving $a_n = -(n-1)a_{n-1} + (n-1)!$ is similar.

Tutorial Exercise T5.3
Solve the following recurrence relation:

$$a_n = n + 1 + \frac{1}{n} \sum_{k=0}^{n-1} a_k \text{ for } n > 0 \text{ and } a_0 = 2$$

Solution
If we apply the repertoire method, we can quickly realize that for $a_n = 1$, $a_n - \frac{1}{n}\sum_{k=0}^{n-1} a_k = 0$, which means that 1 is a solution to the homogeneous part of the recurrence. For $a_n = n$ we obtain $a_n - \frac{1}{n}\sum_{k=0}^{n-1} a_k = (n+1)/2$. To obtain the desired recurrence we take $a_n = 2n+2$, where the first part gives us the inhomogeneous term and the second part accomodates for the initial condition.
Alternatively, one can look at the first few terms, guess the solution to be $2n + 2$ and then prove its correctness by induction.

Homework Exercise H5.1
Solve the following recurrence relation by order reduction:

$$a_0 = 0 \qquad a_1 = 1 \qquad a_{n+2} + a_{n+1} - n^2 a_n = n!$$

This is the same recurrence relation as in T2, but the initial conditions are different. Whereas that exercise asked for a solution using order reduction, for this exercise you can choose whatever method you like.

Solution
We use the repertoire method as we already know the solution for the original recurrence relation. Hence, we have in our repertoire $a'_n = (n-1)!/2$ with values $a'_0 = 0$, $a'_1 = 1/2$ and $f(n) = n!$. Note that we only need to find a solution, which changes the value for a_1. It is therefore sufficient to find some solution to the corresponding homogeneous recurrence, i.e., with $f(n) = 0$.
After some time we try $b_n := (-1)^n(n-1)!/2$. Then $b_1 = 1/2$ and $f(n) = 0$:

$$\begin{aligned} b_{n+2} + b_{n+1} - n^2 b_n = (-1)^{n+2}\frac{(n+1)!}{2} + (-1)^{n+1}\frac{n!}{2} - (-1)^n n^2 \frac{(n-1)!}{2} \\ = \frac{1}{2}(-1)^n\big((n+1)n! - n! - n \cdot n!\big) = 0. \end{aligned}$$

The solution for the recurrence relation with $a_1 = 1$ is $a_n = b_n + a'_n$ because $b_1 = 1/2$ and $a'_1 = 1/2$.

Homework Exercise H5.2

How often is the loop in the following excerpt executed if $0 < i$ holds at the beginning?

```
while i <= j
    i := i+j;
    j :=j+10;
```

Solution

We denote the value of i in the nth repetition by i_n (and similar for j_n). For $i_0 > j_0$, the while-loop is never executed. Let thus $0 < i_0 \leq j_0$. We obtain the recursion

$$\begin{aligned} i_n &= i_{n-1} + j_{n-1} \\ j_n &= j_{n-1} + 10 \end{aligned}$$

which yields (by insertion)

$$\begin{aligned} j_n &= j_0 + 10n \\ i_n &= i_{n-1} + 10(n-1) + j_0 \\ &= i_0 + \sum_{k=1}^{n} (10(k-1) + j_0) \\ &= i_0 + 5n(n-1) + nj_0. \end{aligned}$$

The loop is executed as long as $i_n - j_n \leq 0$, which implies

$$5n^2 + (j_0 - 15)n + i_0 - j_0 \leq 0.$$

We know that for a polynomial of degree two holds

$$ax^2 + bx + c = 0 \iff x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}.$$

For positive n we therefore need

$$n \leq \frac{15 - j_0 + \sqrt{(j_0 - 15)^2 - 20(i_0 - j_0)}}{10} =: a(i_0, j_0)$$

holds. In this case, the loop is hence executed $\lfloor a(i_0, j_0) \rfloor + 1$ times.

Homework Exercise H5.3

Our task is to generate a word of length n over the alphabet $\{0, 1\}$, which contains neither two consecutive zeros nor three consecutive ones.

Daniel proposes the following algorithm: The algorithm generates a word of length n uniformly at random. If the word fulfills the property, it is returned. Otherwise, the algorithm tries again until it finds one.

What is the expected number of rounds the algorithm needs depending on n? It is enough to give the fastest growing term, i.e., something of the form $\alpha f(n) + o(f(n))$. Also, give the expected number of rounds for 32bit words explicitly.

```
0110110110110101   0110110101101101   0110110101010101   0110101101011010   0110101011011011
0110110110110110   0110110101101010   0110110101010110   0110101101011011   0110101011010101
0110110110101101   0110110101101011   0110101101101101   0110101101010101   0110101011010110
0110110110101010   0110110101011010   0110101101101010   0110101101010110   0110101010110101
0110110110101011   0110110101011011   0110101101101011   0110101011011010   0110101010110110
```

0110101010101101	0101011010101010	1101101010110110	1011011011011011	1010110101101010
0110101010101010	0101011010101011	1101101010101101	1011011011010101	1010110101101011
0110101010101011	0101010110110101	1101101010101010	1011011011010110	1010110101011010
0101101101101101	0101010110110110	1101101010101011	1011011010110101	1010110101011011
0101101101101010	0101010110101101	1101011011011010	1011011010110110	1010110101010101
0101101101101011	0101010110101010	1101011011011011	1011011010101101	1010110101010110
0101101101011010	0101010110101011	1101011011010101	1011011010101010	1010101101101101
0101101101011011	0101010101101101	1101011011010110	1011011010101011	1010101101101010
0101101101010101	0101010101101010	1101011010110101	1011010110110101	1010101101101011
0101101101010110	0101010101101011	1101011010110110	1011010110110110	1010101101011010
0101101011011010	0101010101011010	1101011010101101	1011010110101101	1010101101011011
0101101011011011	0101010101011011	1101011010101010	1011010110101010	1010101101010101
0101101011010101	0101010101010101	1101011010101011	1011010110101011	1010101101010110
0101101011010110	0101010101010110	1101010110110101	1011010101101101	1010101011011010
0101101010110101	1101101101101101	1101010110110110	1011010101101010	1010101011011011
0101101010110110	1101101101101010	1101010110101101	1011010101101011	1010101011010101
0101101010101101	1101101101101011	1101010110101010	1011010101011010	1010101011010110
0101101010101010	1101101101011010	1101010110101011	1011010101011011	1010101010110101
0101101010101011	1101101101011011	1101010101101101	1011010101010101	1010101010110110
0101011011011010	1101101101010101	1101010101101010	1011010101010110	1010101010101101
0101011011011011	1101101101010110	1101010101101011	1010110110110101	1010101010101010
0101011011010101	1101101011011010	1101010101011010	1010110110110110	1010101010101011
0101011011010110	1101101011011011	1101010101011011	1010110110101101	
0101011010110101	1101101011010101	1101010101010101	1010110110101010	
0101011010110110	1101101011010110	1101010101010110	1010110110101011	
0101011010101101	1101101010110101	1011011011011010	1010110101101101	

Solution

We say a word is correct if it contains neither two consecutive zeros nor three ones. We count the number of correct words of length n to derive the probability that a random word is correct. With that it is easy to compute the expected number of rounds the algorithm needs.

Let a_n be the number of correct words of length n and b_n the number of words among them that start with a 1. For b_n we get the recurrence $b_n = b_{n-2} + b_{n-3}$ for $n \geq 3$ because such a word can start either with 10 or 110, followed again by a word that starts with a 1. In general a word starts with either with 01 or with 1 giving us the recurrence $a_n = b_{n-1} + b_n$ for $n \geq 2$. As a_n is a linear combination of b_n and b_{n-1} every solution for b_n is also a solution for a_n. Hence, $a_n = a_{n-2} + a_{n-3}$ for $n \geq 4$.

Let us solve this recurrence. We compute the basic cases by hand: $a_0 = 1$, $a_1 = 2$, $a_2 = 3, a_3 = 4, a_4 = 5, a_5 = 7, a_6 = 9$.

The characteristic polynomial is $z^3 - z - 1$ with the roots

$$z_1 = \frac{\frac{\sqrt{3}i}{2} + \frac{-1}{2}}{3\left(\frac{\sqrt{23}}{23^{\frac{3}{2}}} + \frac{1}{2}\right)^{\frac{1}{3}}} + \left(\frac{\sqrt{23}}{23^{\frac{3}{2}}} + \frac{1}{2}\right)^{\frac{1}{3}} \left(\frac{-1}{2} - \frac{\sqrt{3}i}{2}\right),$$

$$z_2 = \frac{\frac{-1}{2} - \frac{\sqrt{3}i}{2}}{3\left(\frac{\sqrt{23}}{23^{\frac{3}{2}}} + \frac{1}{2}\right)^{\frac{1}{3}}} + \left(\frac{\sqrt{23}}{23^{\frac{3}{2}}} + \frac{1}{2}\right)^{\frac{1}{3}} \left(\frac{\sqrt{3}i}{2} + \frac{-1}{2}\right),$$

$$z_3 = \left(\frac{\sqrt{23}}{23^{\frac{3}{2}}} + \frac{1}{2}\right)^{\frac{1}{3}} + \frac{1}{3\left(\frac{\sqrt{23}}{23^{\frac{3}{2}}} + \frac{1}{2}\right)^{\frac{1}{3}}}$$

Finding the right linear combination of basic solutions gives us the closed form in Figure 1, which is almost useless in this complicated form.

The roots of the characteristic polynomial read as follows as approximate decimal fractions:

$$z_1 = -0.5622795120623012i - 0.6623589786223729$$
$$z_2 = +0.5622795120623012i - 0.6623589786223729$$
$$z_3 = 1.324717957244746$$

Only the real root is bigger than one. The absolute values of the other two are smaller than 0.87. Hence, $a_n = \alpha z_3^n + O(0.87^n)$ with $\alpha = 1.678735602594163\ldots$ and $z_3 = 1.324717957244746\ldots$. We can approximate a_{32} as

$$a_{32} = 1.678735602594163 \cdot 1.324717957244746^{32} \approx 13581.003,$$

while the exact value is $a_{32} = 13581$.
As there are 2^32 many binary words of length 32, the probability to find one is only $13581/2^32 \approx .00000316$. As the number of required rounds is geometrically distributed we need about 316248 rounds on average to find the first good word. Unfortunately this is not a very efficient method for long words.

$$a_n = -\frac{\left(\left(5\,3^{\frac{21}{2}}\left(\frac{\sqrt{23}}{2\,3^{\frac{3}{2}}}+\frac{1}{2}\right)^{6}-7\,3^{\frac{21}{2}}\left(\frac{\sqrt{23}}{2\,3^{\frac{3}{2}}}+\frac{1}{2}\right)^{\frac{17}{3}}-20\,3^{\frac{19}{2}}\left(\frac{\sqrt{23}}{2\,3^{\frac{3}{2}}}+\frac{1}{2}\right)^{\frac{16}{3}}+85\,3^{\frac{17}{2}}\left(\frac{\sqrt{23}}{2\,3^{\frac{3}{2}}}+\frac{1}{2}\right)^{\frac{14}{3}}+14\,3^{\frac{19}{2}}\left(\frac{\sqrt{23}}{2\,3^{\frac{3}{2}}}+\frac{1}{2}\right)^{\frac{13}{3}}-29\,3^{\frac{17}{2}}\left(\frac{\sqrt{23}}{2\,3^{\frac{3}{2}}}+\frac{1}{2}\right)^{4}-56\,3^{\frac{15}{2}}\left(\frac{\sqrt{23}}{2\,3^{\frac{3}{2}}}+\frac{1}{2}\right)^{\frac{11}{3}}-29\,3^{\frac{15}{2}}\left(\frac{\sqrt{23}}{2\,3^{\frac{3}{2}}}+\frac{1}{2}\right)^{\frac{10}{3}}+14\,3^{\frac{15}{2}}\left(\frac{\sqrt{23}}{2\,3^{\frac{3}{2}}}\right.}{}$$

$$+\frac{\left(\left(5\,3^{\frac{21}{2}}\left(\frac{\sqrt{23}}{2\,3^{\frac{3}{2}}}+\frac{1}{2}\right)^{6}-7\,3^{\frac{21}{2}}\left(\frac{\sqrt{23}}{2\,3^{\frac{3}{2}}}+\frac{1}{2}\right)^{\frac{17}{3}}-20\,3^{\frac{19}{2}}\left(\frac{\sqrt{23}}{2\,3^{\frac{3}{2}}}+\frac{1}{2}\right)^{\frac{16}{3}}+85\,3^{\frac{17}{2}}\left(\frac{\sqrt{23}}{2\,3^{\frac{3}{2}}}+\frac{1}{2}\right)^{\frac{14}{3}}+14\,3^{\frac{19}{2}}\left(\frac{\sqrt{23}}{2\,3^{\frac{3}{2}}}+\frac{1}{2}\right)^{\frac{13}{3}}-29\,3^{\frac{17}{2}}\left(\frac{\sqrt{23}}{2\,3^{\frac{3}{2}}}+\frac{1}{2}\right)^{4}-56\,3^{\frac{15}{2}}\left(\frac{\sqrt{23}}{2\,3^{\frac{3}{2}}}+\frac{1}{2}\right)^{\frac{11}{3}}-29\,3^{\frac{15}{2}}\left(\frac{\sqrt{23}}{2\,3^{\frac{3}{2}}}+\frac{1}{2}\right)^{\frac{10}{3}}+14\,3^{\frac{15}{2}}\left(\frac{\sqrt{23}}{2\,3^{\frac{3}{2}}}+\frac{1}{2}\right)^{3}+\right.}{3}$$

$$+\frac{\left(\left(\frac{\sqrt{23}}{2\,3^{\frac{3}{2}}}+\frac{1}{2}\right)^{\frac{1}{3}}+\frac{1}{3\left(\frac{\sqrt{23}}{2\,3^{\frac{3}{2}}}+\frac{1}{2}\right)^{\frac{1}{3}}}\right)^{n}\left(1215\left(\frac{\sqrt{23}}{2\,3^{\frac{3}{2}}}+\frac{1}{2}\right)^{\frac{8}{3}}+1701\left(\frac{\sqrt{23}}{2\,3^{\frac{3}{2}}}+\frac{1}{2}\right)^{\frac{7}{3}}+1782\left(\frac{\sqrt{23}}{2\,3^{\frac{3}{2}}}+\frac{1}{2}\right)^{2}+567\left(\frac{\sqrt{23}}{2\,3^{\frac{3}{2}}}+\frac{1}{2}\right)^{\frac{5}{3}}+135\left(\frac{\sqrt{23}}{2\,3^{\frac{3}{2}}}+\frac{1}{2}\right)^{\frac{4}{3}}\right)}{729\left(\frac{\sqrt{23}}{2\,3^{\frac{3}{2}}}+\frac{1}{2}\right)^{4}+1215\left(\frac{\sqrt{23}}{2\,3^{\frac{3}{2}}}+\frac{1}{2}\right)^{\frac{10}{3}}+891\left(\frac{\sqrt{23}}{2\,3^{\frac{3}{2}}}+\frac{1}{2}\right)^{\frac{8}{3}}+378\left(\frac{\sqrt{23}}{2\,3^{\frac{3}{2}}}+\frac{1}{2}\right)^{2}+99\left(\frac{\sqrt{23}}{2\,3^{\frac{3}{2}}}+\frac{1}{2}\right)^{\frac{4}{3}}+15\left(\frac{\sqrt{23}}{2\,3^{\frac{3}{2}}}+\frac{1}{2}\right)^{\frac{2}{3}}+1}$$

Abb. 1: This is the closed formula for a_n.