Exercise for Analysis of Algorithms

Exercise T26

Find the exponential generating function for

$$a_n = a_{n-1} + (n-1)a_{n-2} + (n = 0).$$

Solution:

We transform both sides:

$$\sum_{n=0}^{\infty} a_n \frac{z^n}{n!} = \sum_{n=0}^{\infty} a_{n-1} \frac{z^n}{n!} + \sum_{n=0}^{\infty} (n-1)a_{n-2} \frac{z^n}{n!} + 1$$

Remember the following two rules for EGFs:

$$(1) \qquad \int_0^z A(t)dt = \sum_{n=0}^{\infty} a_{n-1} \frac{z^n}{n!}$$

$$(2) \qquad zA(t) = \sum_{n=0}^{\infty} n a_{n-1} \frac{z^n}{n!}$$

The first summand of the right side can be expressed by rule (1). The second summand can be expressed by first applying rule (1) and then rule (2).

$$A(z) = \int_0^z A(t)dt + \int_0^z \sum_{n=0}^{\infty} n a_{n-1} \frac{t^n}{n!} dt + 1 = \int_0^z A(t)dt + \int_0^z tA(t)dt + 1$$

We differentiate:

$$A'(z) = A(z) + zA(z)$$

We substitute $A(z) = e^{f(z)}$ and solve for $f(z)$.

$$f'(z)e^{f(z)} = e^{f(z)} + ze^{f(z)}$$

$$f'(z) = 1 + z$$

$$f(z) = z + z^2/2 + c$$

This means

$$A(z) = e^{z+z^2/2+c}.$$

For $z = 0$ we have

$$e^c = A(0) = \int_0^0 A(t)dt + \int_0^0 tA(t)dt + 1 = 1$$

which implies c=0 and therefore

$$A(z) = e^{z+z^2/2}.$$

Exercise T27
An algorithm is given an array of length $n \geq 0$ and, if $n \geq 2$, for each $1 \leq k \leq n$ calls itself on some random subarray of length k with probability $\frac{1}{2}$. Compute the exponential growth of the running time of this algorithm.

Solution:
Let t_n be the expected number of calls of the algorithm for an array of length n. Then $t_0 = t_1 = 1$ and $t_n = 1 + \frac{1}{2}\sum_{k=1}^{n} t_k$ for $n > 1$. To get an equation that holds for all n, we let

$$t_n = 1 + \frac{1}{2}\sum_{k=1}^{n} t_k - \frac{1}{2}(n = 1).$$

The corresponding OGF $T(z)$ is

$$T(z) = \frac{1}{1-z} + \frac{1}{2}\frac{1}{1-z}T(z) - \frac{1}{2}z,$$

where we used the convolution

$$\frac{1}{1-z}T(z) = \sum_{n\geq 0} 1z^n \sum_{n\geq 0} t_n z^n = \sum_{n\geq 0}\sum_{k=0}^{n} 1t_k z^n = \sum_{n\geq 0}\sum_{k=1}^{n} t_k z^n = \sum_{n\geq 0} t_n z^n.$$

Solving the equation for $T(z)$ leads to $T(z) = \frac{z^2-z+2}{1-2z}$. Since the dominant singularity is located at $z = \frac{1}{2}$, we get an asympotic running time of $\asymp 2^n$.

Exercise H18
Calculate $[z^n]G(z)$ up to an additive error of $O(4^n)$ for

$$G(z) = \frac{15z^2 + 8z + 1}{15z^2 - 8z + 1}.$$

Solution:
We factor

$$G(z) = \frac{15z^2 + 8z + 1}{(3z-1)(5z-1)}.$$

The singularities are $\frac{1}{3}$ and $\frac{1}{5}$. The dominant singularity is $\frac{1}{5}$. This means

$$[z^n]G(z) = c[z^n]\frac{1}{5z-1} + O(3^n).$$

We observe how $G(z)$ behaves in the limit:

$$\lim_{z\to\frac{1}{5}} \frac{G(z)}{1/(5z-1)} = -8$$

Therefore $c = -8$ and

$$[z^n]G(z) = [z^n]\frac{8}{1-5z} + O(3^n) = 8\cdot 5^n + O(3^n).$$

Exercise H19

$$A(z) = \frac{\sqrt{1 - z^7}}{2z^2 - 3z + 1} \qquad B(z) = \frac{1 - z^2}{e^{z+3z^2}} \qquad C(z) = z^5 + 3z^2(z^3 + z^2 + 8)$$

Sort the coeffcients a_n, b_n und c_n by their asymptotic growth in ascending order.

Solution:

We have $c_n = 0$ except for finitely many exceptions. The sequence b_n grows subexponentially because $B(z)$ has no singularities. The sequence a_n has at least one singularity at $\frac{1}{2}$ and therefore grows exponentially. In the limit we therefore have $c_n < b_n < a_n$.