RWTH Aachen **WS 2017**
Theoretical Computer Science Exercise 10
Rossmanith–Dreier–Kuinke 16.1.2018

Exercise for Analysis of Algorithms

Today's tutorial will be based on the following GF:

$$U(z) := \frac{1 - z - \sqrt{(1-3z)(1+z)}}{2z}.$$

Useful values of the Gamma function:

x	$-\frac{5}{2}$	$-\frac{3}{2}$	$-\frac{1}{2}$	$\frac{1}{2}$	$\frac{3}{2}$	$\frac{5}{2}$
$\Gamma(x)$	$-\frac{8\sqrt{\pi}}{15}$	$\frac{4\sqrt{\pi}}{3}$	$-2\sqrt{\pi}$	$\sqrt{\pi}$	$\frac{\sqrt{\pi}}{2}$	$\frac{3\sqrt{\pi}}{4}$

Exercise T22

Prove that $[z^n]U(z) = 3^n n^{O(1)}$.

Solution:
The dominant singularity is located at 1/3. It is an algebraic singularity, since

$$U(z) = \frac{1-z}{2z} - (1-3z)^{1/2}\frac{\sqrt{1+z}}{2z} =: h(z) + (1-3z)^{1/2}g(z),$$

where $h(z)$ and $g(z)$ are analytic in 1/3. Unfortunately, $h(z)$ is not analytic on the disc $|z| \leq 1/3$ and thus we cannot apply the theorem. To get out of this dilemma, we can use $U'(z) := 2zU(z)$. Here, we know that $U'(z)$ and $U(z)$ only differ by an index-shift and a constant factor of two. Now, $U'(z)$ can be written as:

$$U'(z) = 1 - z - (1-3z)^{1/2}\sqrt{1+z} =: h'(z) + (1-3z)^{1/2}g'(z).$$

$U'(z)$ has the same dominant singularity $\alpha = 1/3$, but $h'(z)$ and $g'(z)$ are analytic on $|z| \leq 1/3$. The singularity has order $c = 1/2$. Thus we can apply the theorem, which yields that $[z^n]U'(z) = 3^n n^{O(1)}$. Even better, we can estimate the coefficients as

$$[z^n]U'(z) = -\frac{g'(\alpha)n^{-c-1}}{\Gamma(-c)\alpha^n} + o(3^n n^{-1/3-1}) = \frac{\sqrt{1+1/3}}{2\sqrt{\pi}}3^n n^{-3/2} + o(3^n n^{-3/2}) = \Theta(n^{-3/2}3^n).$$

Exercise T23

Let $V(z) := -\sqrt{(1-3z)(1+z)}$. What is the relation of $[z^n]V(z)$ and $[z^n]U(z)$?

Solution:
Since $V(z) = 2zU(z) - 1 + z$ we have

$$[z^n]V(z) = 2[z^{n-1}]U(z) - (n = 0) + (n = 1).$$

In other words, setting $v_n = [z^n]V(z)$ and $u_n = [z^n]U(z)$:

$$v_n = \begin{cases} 2u_{n-1} & \text{falls } n > 1 \\ 2u_0 + 1 & \text{falls } n = 1 \\ -1 & \text{falls } n = 0 \end{cases}$$

Testing these with the maxima tool confirms the formula for small values:

```
(C1) V(z):=-sqrt((1-3*z)*(1+z));

(D1)                     V(z) := - SQRT((1 - 3 z) (1 + z))
(C2) U(z):=(1-z+V(z))/(2*z);


                                    1 - z + V(z)
(D2)                        U(z) := ------------
                                        2 z
(C3) taylor(U(z),z,0,9);

               2       3       4       5        6        7         8         9
(D3)/T/ z + z   + 2 z   + 4 z   + 9 z   + 21 z   + 51 z   + 127 z   + 323 z   + . . .

(C4) taylor(V(z),z,0,8);

                         2       3       4       5        6        7         8
(D4)/T/ - 1 + z + 2 z   + 2 z   + 4 z   + 8 z   + 18 z   + 42 z   + 102 z   + . . .
```

Exercise T24
Compute $W(z) := V(z) + 2/\sqrt{3}\,\sqrt{1-3z}$ and estimate $[z^n]W(z)$ *as good as possible.*

Solution:

$$W(z) = 2/\sqrt{3}\,\sqrt{1-3z} - \sqrt{1-3z}\sqrt{1+z} = \sqrt{1-3z}(2/\sqrt{3} - \sqrt{1+z}).$$

Using $h(z) = 0$ and $g(z) = 2/\sqrt{3} - \sqrt{1+z}$, we could already apply the theorem. Unfortunately, $g(1/3) = 0$, and thus the theorem only yields the rather bad estimation $[z^n]W(z) = o(3^n n^{-3/2})$. We therefore rewrite $W(z)$ as

$$W(z) = (1-3z)^{3/2}\,\frac{2/\sqrt{3} - \sqrt{1+z}}{1-3z} =: (1-3z)^{3/2} g(z).$$

Since $g(z)$ is analytic on $|z| \le 1/3$, $1/3$ is a singularity of order $3/2$ for the whole expression and we can apply the theorem to get a better estimation of $[z^n]W(z)$ (using L'Hospital for $g(1/3)$):

$$\lim_{z\to\frac{1}{3}} g(z) = \lim_{z\to\frac{1}{3}} \frac{2/\sqrt{3} - \sqrt{1+z}}{1-3z} = \lim_{z\to\frac{1}{3}} \frac{1}{6\sqrt{1+z}} = \frac{\sqrt{3}}{12}$$

$$[z^n]W(z) = \frac{\sqrt{3}}{12}\frac{n^{-5/2}3^n}{\Gamma(-3/2)} + o(3^n n^{-5/2}) = \frac{\sqrt{3}}{16\sqrt{\pi}}\frac{3^n}{n^{5/2}} + o(3^n n^{-5/2})$$

For your convenience, here are some number that show the quality of above estimation: The exact value for $[z^{50}]W(z)$ is 2568001833006140416, while the above estimation yields approx. 2.480×10^{18}.

Exercise T25
What follows for $[z^n]V(z)$?

Solution:

$$[z^n]V(z) = -[z^n]\frac{2}{\sqrt{3}}\sqrt{1-3z} + [z^n]W(z) = -\frac{2}{\sqrt{3}}\binom{1/2}{n}(-3)^n + \frac{\sqrt{3}}{16\sqrt{\pi}}\frac{3^n}{n^{5/2}} + o(3^n n^{-5/2})$$

By the homework assignment below, we may also write this as

$$[z^n]V(z) = -\frac{2}{\sqrt{3}\Gamma(-1/2)}\frac{3^n}{n^{3/2}} + \frac{\sqrt{3}}{16\sqrt{\pi}}\frac{3^n}{n^{5/2}} + o(3^n n^{-5/2}).$$

Exercise H15
Prove that

$$\binom{-r}{n} = (-1)^n\binom{r+n-1}{n}$$

for $r \in \mathbf{R}, n \in \mathbf{Z}$.

Solution:
We have

$$\begin{aligned}\binom{-w}{n} &= \frac{(-w)^{\underline{n}}}{n!} \\ &= \frac{(-w)(-w-1)\cdots(-w-n+1)}{n!} \\ &= \frac{(-1)^n w(w+1)\cdots(w+n-1)}{n!} \\ &= \frac{(-1)^n (n+w-1)^{\underline{n}}}{n!} \\ &= (-1)^n\binom{n+w-1}{n}\end{aligned}$$

Exercise H16
Prove that

$$[z^n](1-z)^w \sim \frac{n^{-w-1}}{\Gamma(-w)}$$

for $w \in \mathbf{C}$ without using the theorem of the lecture. (The idea of this assignment is to get a deeper insight into the theorem.)

Hint: Use Newton's formula, then the first exercise on this sheet. Now replace the binomial coefficient by factorials or the gamma function. In the first case, you need to be careful with a definition of factorials for real numbers. In general, however, $\Gamma(n+1) = n!$.

Solution:

$$\begin{aligned}
[z^n](1-z)^w &= \binom{w}{n}(-1)^n \\
&= \binom{n-w-1}{n} \\
&= \frac{(n-w-1)!}{n!(-w-1)!} \\
&= \frac{1}{n^{\underline{w+1}}(-w-1)!} \\
&= \frac{1}{n^{\underline{w+1}}\Gamma(-w)} \\
&= \frac{n^{-w-1}}{\Gamma(-w)}\Big(1+O\Big(\frac{1}{n}\Big)\Big)
\end{aligned}$$

Exercise H17

Determine $[z^n]U(z)$ in the form $h_n + o(3^n n^{-7/2})$, where h_n shall be in closed form.

Solution:

In Exercise 24 we had

$$W(z) = (1-3z)^{3/2}\frac{2/\sqrt{3} - \sqrt{1+z}}{1-3z}.$$

To solve this Excercise however we need to do two steps instead of one. For $z \to \frac{1}{3}$ holds, according to Exercise 24,

$$W(z) \sim (1-3z)^{3/2}\frac{1}{6\sqrt{1+\frac{1}{3}}} = (1-3z)^{3/2}\frac{\sqrt{3}}{12}.$$

As a second function we set

$$\begin{aligned}
R(z) &= W(z) - (1-3z)^{3/2}\frac{\sqrt{3}}{12} \\
&= (1-3z)^{5/2}\Big(\frac{2/\sqrt{3}-\sqrt{1+z}}{(1-3z)^2} - \frac{\sqrt{3}}{12}\frac{1}{1-3z}\Big) \\
&= (1-3z)^{5/2}\frac{2/\sqrt{3}-\sqrt{1+z}-\sqrt{3}/12(1-3z)}{(1-3z)^2}.
\end{aligned}$$

Now we can use the L'Hospital-Rule, but because we did two steps above need to apply it two times as well. The first and second derivative of the denominator $(1-3z)^2$ are $-6(1-3z)$ and 18. For the numerator we use `maxima`; the following input yields the second derivative:

```
f(z) := 2/sqrt(3) - sqrt(1+z) - sqrt(3)/12 * (1-3*z);
diff(f(z),z,2);
```

The output is $1/(4*(4/3)^{3/2})$, the expression in brackets is at $\frac{1}{3}$ therefore

$$\frac{1}{72}\left(\frac{4}{3}\right)^{-\frac{3}{2}}.$$

The Theorem of Darboux yields

$$[z^n]R(z) \sim \frac{\sqrt{27}}{72 \cdot 8} \frac{n^{-7/2} 3^n}{\Gamma(-\frac{5}{2})}.$$

With $W(z) = R(z) + (1-3z)^{3/2}$ we can now find an approximation of $W(z)$ as well:

$$[z^n]W(z) \sim \frac{\sqrt{27}}{576} \frac{n^{-7/2} 3^n}{\Gamma(-\frac{5}{2})} + \frac{\sqrt{3}}{12}(-1)^n \binom{3/2}{n} 3^n.$$

With $V(z) = \frac{2}{\sqrt{3}}\sqrt{1-3z} + W(z)$ we can approximate $V(z)$. Here we want to express it in O-Notation, to get rid of the $\sim$-Symbol:

$$[z^n]V(z) = \frac{\sqrt{27}}{576} \frac{n^{-7/2} 3^n}{\Gamma(-\frac{5}{2})} + \frac{\sqrt{3}}{12}(-1)^n \binom{3/2}{n} 3^n + \frac{2}{\sqrt{3}}(-1)^n \binom{1/2}{n} 3^n + o(3^n n^{-7/2}).$$

To get a value for $U(z)$ from $[z^n]V(z)$ is straight- forward via Exercise 23.