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Theoretical Computer Science
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Exercise 9
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Exercise for Analysis of Algorithms

Exercise T20

In this exercise we consider the following (regular) CFG G:

$$S \to abA \mid bS \mid a$$
$$A \to bA \mid aS$$

1. Find a generating function for number of words s_n in $L(G)$ that have length n.
2. What is the dominant singularity and what kind of singularity is it?
3. What is the exponential growth of s_n?
4. How precisely can you estimate s_n with just the knowledge of the dominating singularity and its nature?
5. Find a closed formula for s_n with an additive error of at most $O(0.8^n)$.

Solution:

1. Since the grammar is unique, the symbolic method gives us

$$S(z) = z^2 A(z) + zS(z) + z,$$

$$A(z) = zA(z) + zS(z).$$

We solve the latter for $A(z)$ and obtain $A(z) = zS(z)/(1-z)$. Now we can insert it into the former. This yields

$$S(z) = z^3 S(z)/(1-z) + zS(z) + z.$$

We then solve for $S(z)$ and get the generating function

$$S(z) = \frac{z}{1 - z - z^3/(1-z)} = \frac{z(1-z)}{(1-z)^2 - z^3}.$$

2. The singularities are the roots of of the denominator. We ask Maxima

`solve((1-z)^2-z^3,z)` and get

$$z = -\frac{\left(9\sqrt{23}+11\sqrt{3}\right)^{\frac{2}{3}}\left(\sqrt{3}\,i+1\right)+5\,2^{\frac{2}{3}}\,3^{\frac{5}{6}}\,i-2^{\frac{4}{3}}\,3^{\frac{1}{6}}\left(9\sqrt{23}+11\sqrt{3}\right)^{\frac{1}{3}}-5\,2^{\frac{2}{3}}\,3^{\frac{1}{3}}}{2^{\frac{4}{3}}\,3^{\frac{7}{6}}\left(9\sqrt{23}+11\sqrt{3}\right)^{\frac{1}{3}}},$$

$$z = \frac{\left(9\sqrt{23}+11\sqrt{3}\right)^{\frac{2}{3}}\left(\sqrt{3}\,i-1\right)+5\,2^{\frac{2}{3}}\,3^{\frac{5}{6}}\,i+2^{\frac{4}{3}}\,3^{\frac{1}{6}}\left(9\sqrt{23}+11\sqrt{3}\right)^{\frac{1}{3}}+5\,2^{\frac{2}{3}}\,3^{\frac{1}{3}}}{2^{\frac{4}{3}}\,3^{\frac{7}{6}}\left(9\sqrt{23}+11\sqrt{3}\right)^{\frac{1}{3}}},$$

$$z = \frac{\left(9\sqrt{23}+11\sqrt{3}\right)^{\frac{2}{3}}+2^{\frac{1}{3}}\,3^{\frac{1}{6}}\left(9\sqrt{23}+11\sqrt{3}\right)^{\frac{1}{3}}-5\,2^{\frac{2}{3}}\,3^{\frac{1}{3}}}{2^{\frac{1}{3}}\,3^{\frac{7}{6}}\left(9\sqrt{23}+11\sqrt{3}\right)^{\frac{1}{3}}}$$

Wolfram Alpha even gives us a nice diagram

`http://www.wolframalpha.com/input/?i=(1-z)%5E2-z%5E3+%3D+0`

from which we see that we have a small real and two larger complex conjugated singularities. We evaluate them numerically and see that their magitude are $\beta \approx 1.32471$ and $\alpha \approx 0.5698402909980533$. The dominant singularity is α. We decomposed the denominator of $S(z)$ into three roots of degree one. The function $S(z)(z-\alpha)$ is therefore analytical at α. This means that α is a pole of first order.

For a meromorph generating function $B(z)$ with poles $\alpha_1, \dots, \alpha_m$ we know that there exist polynomials $P_1(n), \dots, P_m(n)$ such that

$$[z^n]B(z) = \sum_{j=1}^{m} P_j(n)\alpha_j^n$$

and the degree of the polynomial $P_j(n)$ is one smaller than the order of the pole α_j. Since we have three poles of first order the polynomials are constants. We have

$$s_n = c_1(\alpha^{-n}) + c_2(\beta^{-n}).$$

for some constants c_1 and c_2.

3. The exponetial growth is $\alpha^{-n} \approx 1.754877666246692655^n$.

4. Notice that $c_2(\beta^{-n}) = O(0.8^n)$. If we can find the hidden factor c_1 we get a good approximation with vanishing additive error for s_n. We can decompose

$$S(z) = \sum_{n=1}^{\infty} c_1\alpha^{-n}z^n + c_2\beta^{-n}z^n = \frac{c_1}{1-z/\alpha} + \frac{c_2}{1-z/\beta}$$

Let

$$B(z) = \frac{1}{1-z/\alpha} \quad \text{and} \quad E(z) = \frac{1}{1-z/\beta}$$

Notice that $\lim_{z\to\alpha} B(z) = \infty$ while $\lim_{z\to\alpha} E(z)$ is constant. Then

$$\lim_{z\to\alpha}\frac{S(z)}{B(z)} = \lim_{z\to\alpha}\frac{c_1B(z)+c_2E(z)}{B(z)} = c_1.$$

We use maxima to approximate

$$\lim_{z\to\alpha}\frac{S(z)}{B(z)} = \lim_{z\to\alpha}\frac{z(1-z)(1-z/\alpha)}{(1-z)^2-z^3} \approx 0.23448675.$$

We use the following programm to verify the correctness

```
s = range(0,1000)
a = range(0,1000)
s[0] = 0
s[1] = 1
a[0] = 0
a[1] = 0
for n in range(2, 100):
    s[n] = a[n-2]+s[n-1]
        a[n] = a[n-1]+s[n-1]
            print n, s[n], 1.0*s[n]/s[n-1],
            0.23448675*1.754877666246692655**n/s[n]
```

The last line states

```
99 355268071453933228439241 1.75487766625 0.999999931817.
```

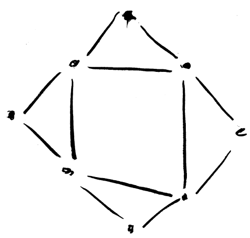
Indeed, after 100 iterations we only make a multiplicative error of 0.999999931817.

Exercise T21

An algorithm I computes an optimal independent set for an undirected graph $G = (V, E)$ of size n as follows: It picks a vertex v with maximal degree. If this degree is at most two, then the graph is a collection of cycles and paths and the solution is computed in linear time.

Otherwise, the optimal independent set either contains v (and then cannot contain any vertex in $N(v)$) or it does not. Hence, the algorithm recursively computes the two independent sets $I(G[V - N(v)])$ and $I(G[V - \{v\}])$ and then chooses the bigger one, or the first if they have the same size.

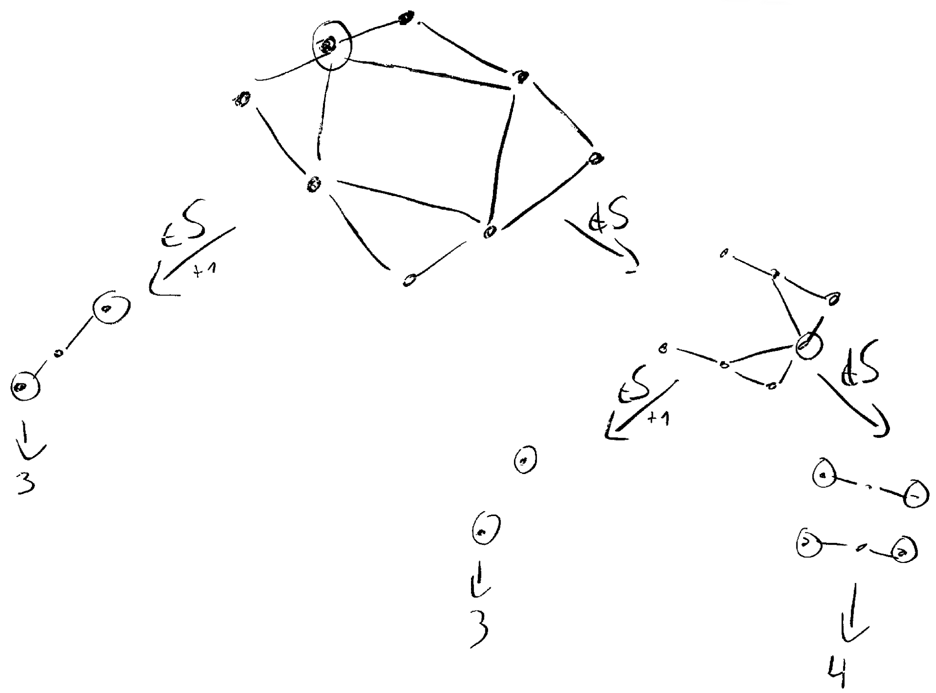
1. Simulate the algorithm on this graph:



2. Estimate its asymptotic running time up to a constant factor.

Solution:

1.



The best solution is therefore obtained by picking the 4 corner points.

2.

The smallest reduction happens if the graph has max degree 3, in this case we get the following recurrence relation:

$$t(n) = t(n-4) + t(n-1) + (n = 0)$$

We find a generating function:

$$\begin{aligned} S(z) &= z^4 S(z) + zS(z) + 1 \\ (1 - z^4 - z)S(z) &= 1 \\ S(z) &= \frac{1}{1 - z^4 - z} \end{aligned}$$

Finding the dominant singularity α of $S(z)$ by setting $1 - z^4 - z = 0$, yields the numerical result $\alpha \approx 0.72449$. And a runtime of the algorithm of $O(\alpha^{-n}) = O(1.38028^n)$.

Exercise H14

In this exercise we will look at 2-3-trees. They are rooted, ordered trees. Each internal node has either two or three children. As usual, the size of a 2-3-tree will be the number of its internal nodes.

1. How can you define 2-3-trees recursively?
2. Enumerate all 2-3-trees of size two. How many are there? How many trees exist of sizes zero and one?
3. Find a generating function $Q(z)$ for the number q_n of 2-3-trees with size n.
4. What is the dominant singularity of $Q(z)$ and what is the exponential growth of q_n? Use a computer algebra system. Do not give up when you see horrifying formulas.

Solution:
We get immeadiately the equation $Q(z) = zQ(z)^2 + zQ(z)^3 + 1$. If we solve it with the help of a computer algebra system, we get a very messy result, but we can spot the expression $\sqrt{z^2 + 11z - 1}$, which defines one the singularities. It seems that it is probably the dominant one. If that is true, then $\alpha = (5^{3/2} - 11)/2$ is the singularity for which we are looking. The exponential growth is then $\alpha^{-n} \approx 11.09016994374933^n$.

A closer look shows that this is indeed the dominant singularity.