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Theoretical Computer Science
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Exercise 7
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Exercise for Analysis of Algorithms

Exercise T14

Use the symbolic method to calculate the number of words of length n that can be created by the following grammar:

$$P \to \quad \text{😀} P \text{😍} \quad | \quad \text{🤡} P \text{🙀} \quad | \quad \text{💩} \quad | \quad \text{💩} P$$

Solution:

The language of this grammar can be described by the following recursive definition.

$$P = (\{\text{😀}\} \times P \times \{\text{😍}\}) \cup (\{\text{🤡}\} \times P \times \{\text{🙀}\}) \cup \{\text{💩}\} \cup (\{\text{💩}\} \times P)$$

We need to define the weight of the atomic elements.

$$|\text{😀}| = |\text{😍}| = |\text{🤡}| = |\text{🙀}| = |\text{💩}| = 1$$

Let $T(z)$ be the generating function for the number of words of length n generated by the grammar. The symbolic method yields

$$T(z) = 2z^2T(z) + z + zT(z).$$

We transform this into

$$T(z) = \frac{z}{1 - z - 2z^2}.$$

Notice that $1 - z - 2z^2 = (z+1)(1-2z)$. We want to find a partial fraction decomposition of the form

$$\frac{z}{1 - z - 2z^2} = \frac{A}{z+1} + \frac{B}{1-2z}.$$

By setting $z = 0$ we get $0 = A + B$ and by setting $z = 1$ we get $-1/2 = A/2 - B$. Together they yield $A = -1/3$ and $B = 1/3$. This means

$$T(z) = -\frac{1}{3}\frac{1}{z+1} + \frac{1}{3}\frac{1}{1-2z}$$

and therefore $[z^n]T(z) = \frac{1}{3}(2^n - (-1)^n)$. There are $\frac{1}{3}(2^n - (-1)^n)$ words of length n.

Exercise T15

Find a bivariate generating fucntion and a closed-form expression for the number of bitstrings of length n that contain exactly m ones and do not contain the substring 11.

Solution:
The bitstrings which do not contain 11 as substring are generated can be described by the recursive definition $F = 0F + 10F + \varepsilon + 1$. We get the following generating function:

$$\begin{aligned}
F(z) &= zF(z) + uz^2F(z) + 1 + uz \\
&= \frac{1+uz}{1-z-uz^2} \\
&= (1+uz)\sum_{k\geq 0} z^k(1+uz)^k \\
&= (1+uz)\sum_{k\geq 0} z^k \sum_{i=0}^{k} \binom{k}{i} u^i z^i \\
&= (1+uz)\sum_{k\geq 0} \sum_{i=0}^{k} \binom{k}{i} u^i z^{i+k} \\
&= \sum_{k\geq 0} \sum_{i=0}^{k} \binom{k}{i} u^i z^{i+k} + \sum_{k\geq 0} \sum_{i=0}^{k} \binom{k}{i} u^{i+1} z^{i+k+1}
\end{aligned}$$

We are interested in the coefficients corresponding to $z^n u^m$. Thus we need to consider the summand with $i = m \wedge i + k = n$ (front) and $i + 1 = m \wedge i + k + 1 = n$ (back). This gives the closed formula for the number of valid bitstrings:

$$\binom{n-m}{m} + \binom{n-m}{m-1} = \binom{n-m+1}{m}$$

Exercise H10
Use the symbolic method to calculate the number of words of length n that can be created by the following grammar:

$$\begin{aligned}
Q &\rightarrow \text{😀} P \text{😍} \\
P &\rightarrow PQ \mid \varepsilon
\end{aligned}$$

Hint: Use the sequence operator.

Solution:
The language of this grammar can be described by the recursive definition

$$Q = (\{\text{😀}\} \times \bigcup_{i=0}^{\infty} Q^i \times \{\text{😍}\}).$$

The symbolic method yields

$$Q(z) = \frac{z^2}{1 - Q(z)}.$$

We solve the quadratic equation

$$Q(z)^2 - Q(z) + z^2 = 0.$$

Only the positive solution is feasible. Therefore

$$Q(z) = \frac{1}{2} + \frac{\sqrt{1-4z^2}}{2} = \frac{1}{2} + \frac{1}{2}(1-4z^2)^{\frac{1}{2}} = \frac{1}{2} + \frac{1}{2}\sum_{n}^{\infty}\binom{1/2}{n}(-1)^n 4^n z^{2n}.$$

This means the number of words of odd length is 0 and the number of words of length $2n$ is $\frac{1}{2}\binom{1/2}{n}(-1)^n 4^n + \frac{1}{2}(n=0)$.

Exercise H11

Find a generating function for the number of trees with exactly n internal and m external vertices $T_{n,m}$. For what values of n, m do we have $T_{n,m} = T_{m,n}$?
Hint: Do not do all the computations by hand. Seek the help of a computer algebra system. `maxima` can solve quadratic equations and can find the coefficients of a generating function via Taylor expansion.

Solution:

We consider a recursive definition of trees. A tree is either an external node or an internal node with at least one subtree. This yields

$$T = \square \cup \bigcirc \times T \times T^*$$

und and the generating function

$$T(u,z) = u + z\frac{T(u,z)}{1-T(u,z)},$$

where u is the number of external and z is the number of internal nodes. We need to solve this equation for $T(u,z)$. We delegate this task to `maxima`: The call `solve(T=u+z*T/(1-T),T)` yields

$$-\frac{\sqrt{z^2+(-2*u-2)*z+u^2-2*u+1}+z-u-1}{2}$$

and

$$+\frac{\sqrt{z^2+(-2*u-2)*z+u^2-2*u+1}-z+u+1}{2}.$$

We know that for $u = z = 0$ the solution needs to be 0. Thus only the first solution is correct.
We get the coefficients by doing a Taylor expansion of $T(u,z)$. We enter

```
T:-((sqrt(z^2+(-2*u-2)*z+u^2-2*u+1)+z-u-1)/2);
taylor(T,[z,u],0,5);
```

into `maxima`. We read the coefficients:

$n+m$	Term
1	u
2	uz
3	$uz^2 + u^2z$
4	$uz^3 + 3u^2z^2 + u^3z$
5	$uz^4 + 6u^2z^3 + 6u^3z^2 + u^4z$

We also want to find out for which values holds $T(u,z) = T(z,u)$. We do so by calculating $T(u,z) - T(z,u)$. We type into `maxima`:

```
TT(u,z):=-((sqrt(z^2+(-2*u-2)*z+u^2-2*u+1)+z-u-1)/2);
ev(TT(u,z)-TT(z,u),expand);
```

The answer is $u - z$. Therefore, the generating function of the difference is $u - z$. The all coefficients of this function are zero except for the case $u = 1, z = 0$ or $u = 0, z = 1$. This means $T_{u,z} = T_{z,u}$ for all other values. This makes sense as there is exactly one tree with a single external node and zero trees with a single internal node.