Exercise for Analysis of Algorithms

Exercise T12

Compute the generating functions of the following series:

1. $a_n = 2^n + 3^n$ 2. $b_n = (n+1)2^{n+1}$ 3. $c_n = \alpha^n \binom{k}{n}$
4. $d_n = n - 1$ 5. $e_n = (n+1)^2$

Solution:

1. The generating function of (α^n) is $\sum_{n \geq 0} \alpha^n z^n$, which yields $\frac{1}{1-\alpha z}$ in closed form. The generating function of $a_n = 2^n + 3^n$ is thus simply $\frac{1}{1-2z} + \frac{1}{1-3z}$.
2. We start with (2^n) and $\frac{1}{1-2z}$. Derivating yields $b_n = (n+1)2^{n+1}$ with generating function $\frac{2}{(1-2z)^2}$.
3. The series $\binom{k}{n}$ has the generating function $(1+z)^k$. Scaling with α results in $c_n = \alpha^n \binom{k}{n}$ with corresponding generating function $(1 + \alpha z)^k$.
4. We already know that the series $(n+1) = 1, 2, 3, 4, \ldots$ belongs to the generating function $\frac{1}{(1-z)^2}$. In order to obtain $d_n = -1, 0, 1, 2, 3, \ldots$, we first shift this twice to the right. This yields $0, 0, 1, 2, 3, 4, \ldots$ with generating function $\frac{z^2}{(1-z)^2}$. Now we subtract $1, 0, 0, \ldots$ and obtain d_n with generating function $\frac{z^2}{(1-z)^2} - 1$.
5. Recall that $(n+1) = 1, 2, 3, 4, \ldots$ has the generating function $\frac{1}{(1-z)^2}$. We shift to the right and obtain (n) as well as $\frac{z}{(1-z)^2}$. Derivating yields the desired series $e_n = (n+1)^2$ with generating function $\frac{z+1}{(1-z)^3}$.

Exercise T13

Compute:

(a) $[z^n]\dfrac{1}{1+2z}$ (b) $[z^n]\dfrac{z+1}{z-1}$ (c) $[z^n]\left(\dfrac{z+1}{z-1}\right)^2$ (d) $[z^n]\dfrac{1}{\sqrt[3]{5+z}}$

Solution:

a)
$\sum_{n \geq 0} \alpha^n z^n$, yields $\frac{1}{1-\alpha z}$. So $[z^n]\frac{1}{1+2z} = (-2)^n$.

b)
We use that

$$\frac{n+1}{n-1} = 1 - 2\frac{1}{1-n}$$

and get $A(z) - 2B(z)$ with

$$[z^n]A(z) = (n = 0)$$

and

$$[z^n]B(z) = 1$$

to obtain $[z^n]\frac{z+1}{z-1} = (n = 0) - 2$.

c)
The convolution rule yields

$$A(z)^2 = \sum_{n=0}^{\infty} \Big(\sum_{k=0}^{n} a_k a_{n-k} \Big) z^n$$

We use our solution for a_n from b) to obtain For $n \neq 0$

$$\sum_{k=0}^{n} a_k a_{n-k} = \Big(\sum_{k=1}^{n-1} a_k a_{n-k} \Big) + a_0 \cdot a_n + a_n \cdot a_0 = 4(n-1) + 4 = 4n$$

and for $n = 0$

$$\sum_{k=0}^{n} a_k a_{n-k} = 1$$

Which yields $[z^n]\left(\frac{z+1}{z-1}\right)^2 = 4n + (n = 0)$

d) We start with

$$\frac{1}{\sqrt[3]{5+z}} = \frac{1}{\sqrt[3]{5}} \frac{1}{\sqrt[3]{1+z/5}}$$

We use that

$$[z^n](1+z)^r = \binom{r}{n}$$

Finally we use scaling with 1/5 to obtain

$$[z^n]\frac{1}{\sqrt[3]{5+z}} = \frac{1}{\sqrt[3]{5}} \binom{-\frac{1}{3}}{n} 5^{-n}$$

Exercise T14

```
f(int n){
  int s=0;
  if (n==0) return 1;
  for (int i=0;i<n;i++)
    s+=f(i);
  return s;
}
```

Compute how often the 5th line of this program is executed using generating functions.

Solution:
Let A_n denote the how often the 5th line is called on input n. We immediately obtain $A_0 = 0$ and $A_n = n + \sum_{i=0}^{n-1} A_i$.
The corresponding generating function is thus

$$A(z) = \frac{z}{(1-z)^2} + \frac{zA(z)}{1-z}.$$

This implies

$$A(z)(1-2z) = \frac{z}{1-z}$$

and therefore

$$\begin{aligned} A(z) &= \frac{1}{(1-z)(1-2z)} z \\ &= \left(\frac{A}{1-z} + \frac{B}{1-2z} \right) z \end{aligned}$$

for some A, B. Setting $z = 0$ implies $A + B = 1$ and $z = -1$ implies $A/2 + B/3 = 1/6$. We easily obtain $A = -1$ and $B = 2$ and therefore

$$A(z) = \left(\frac{2}{1-2z} - \frac{1}{1-z} \right) z.$$

Since the multiplication with z is just a shift, the solution is therefore

$$a_{n+1} = 2 \cdot 2^n - 1 = 2^{n+1} - 1$$

Exercise H9
Solve this recurrence using generating functions:

$$a_n = 2a_{n-1} + 3a_{n-2}$$

and $a_0 = 0$, $a_1 = 2$.

Solution:
We start with $S(z) = \sum_{n\geq 0} a_n z^n$. Since the values a_0 and a_1 are given, we rewrite this as

$$S(z) = a_0 + a_1 z + \sum_{n\geq 2} a_n z^n.$$

Using the recursive definition, this yields

$$S(z) = a_0 + a_1 z + \sum_{n\geq 2} (2a_{n-1} + 3a_{n-2}) z^n = a_0 + a_1 z + \sum_{n\geq 2} 2a_{n-1} z^n + \sum_{n\geq 2} 3a_{n-2} z^n.$$

Shifting results in

$$\begin{aligned} S(z) &= a_0 + a_1 z + z \sum_{n\geq 1} 2a_n z^{n+1} + \sum_{n\geq 0} 3a_n z^{n+2} \\ &= a_0 + a_1 z - 2a_0 z + 2z \sum_{n\geq 0} a_n z^n + 3z^2 \sum_{n\geq 0} a_n z^n \\ &= 2z + 2zS(z) + 3z^2 S(z) \end{aligned}$$

This implies

$$S(z) \quad = \quad \frac{2z}{1-2z-3z^2} = \frac{2z}{(1+z)(1-3z)}$$

Now, we need to find a and b such that

$$S(z) = \frac{2z}{(1+z)(1-3z)} = \frac{a}{1+z} + \frac{b}{1-3z}.$$

Setting $z = 0$ and $z = 1$ implies $a + b = 0$ as well as $-2a + 2b = 2$, and thus $a = -1/2$ and $b = 1/2$. We obtain

$$S(z) = -\frac{1/2}{1+z} + \frac{1/2}{1-3z} = -\frac{1}{2}\sum_{n\geq 0}(-1)^n z^n + \frac{1}{2}\sum_{n\geq 0} 3^n z^n$$

Thus, we have

$$a_n = \frac{1}{2}3^n - \frac{1}{2}(-1)^n.$$