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Theoretical Computer Science
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WS 2017
Exercise 3
21.11.2017

Exercise for Analysis of Algorithms

Exercise T7

Given an array a of length n, an algorithm compares all pairs $(a[i], a[j])$ for all $i < j \le n$, and then calls itself recursively on all proper prefixes of a.
How often does the algorithm compare two pairs? Use the repertoire method!

Solution:

The recurrence is $R_0 = 0$ and

$$R_n = \binom{n}{2} + \sum_{k=0}^{n-1} R_k$$

for $n \ge 1$. Testing a couple of values for R_n, we obtain the repertoire:

R_0	R_n	$f_n = R_n - \sum_{k=0}^{n-1} R_k$
1	a^n	$\frac{2-a}{1-a} a^n - \frac{1}{1-a}$
1	2^n	1
1	1	$1 - n$
0	n	$n - \binom{n}{2}$

To get $\binom{n}{2}$, we can use the last line. To get rid of the linear and constant factors, we use the third and finally the second line, and obtain

$$-\left(n - \binom{n}{2}\right) - (1 - n) + 1 = \binom{n}{2}.$$

Fortunately, this also holds for $R_0 = 0$, and the solution is

$$R_n = 2^n - n - 1.$$

Exercise T8

Solve the following recurrence: Let $a_0 = 1$, $a_1 = 1$, $a_2 = 4$ and

$$a_n = 2a_{n-1} - a_{n-2} + 2a_{n-3}, \text{ for } n \ge 3.$$

Solution:
The characteristic polynomial is z^3-2z^2+z-2. This can be factorized as $(z^2+1)(z-2)$, which gives the set of roots as $\{\pm i, 2\}$. The general solution is thus:

$$a_n = \alpha \cdot 2^n + \beta \cdot (-i)^n + \gamma \cdot i^n.$$

Substituting the values for $n = 0, 1, 2$, we obtain the following system of linear equations:

$$\alpha + \beta + \gamma = 1 \tag{1}$$
$$2\alpha - i\beta + i\gamma = 1 \tag{2}$$
$$4\alpha - \beta - \gamma = 4 \tag{3}$$

Solving this system yields $\alpha = 1$, $\beta = -i/2$ and $\gamma = i/2$. Note that β and γ are complex conjugates. The solution is thus:

$$a_n = 2^n - \frac{i}{2} \cdot (-i)^n + \frac{i}{2} i^n \tag{4}$$
$$= 2^n + (1 + (-1)^{n+1}) \cdot \frac{i^{n+1}}{2}. \tag{5}$$

This solution can be written in a much nicer form using Euler's formula:

$$e^{i\theta} = \cos\theta + i \sin\theta.$$

We may write $i^{n+1}/2$ as follows:

$$\frac{i^{n+1}}{2} = \frac{1}{2}\left(\cos\frac{\pi}{2} + i \sin\frac{\pi}{2}\right)^{n+1} = \frac{e^{\frac{i\pi}{2}\cdot(n+1)}}{2}.$$

Similarly, $(-1)^{n+1} i^{n+1}/2$ may be written as $\frac{1}{2} \cdot e^{-\frac{i\pi}{2}\cdot(n+1)}$. Now note that

$$\frac{e^{\frac{i\pi}{2}\cdot(n+1)} + e^{-\frac{i\pi}{2}\cdot(n+1)}}{2} = \cos\left(\frac{\pi(n+1)}{2}\right).$$

Thus $a_n = 2^n + \cos\frac{\pi(n+1)}{2}$.

Exercise H4
Solve the following recurrence: Let $a_0 = 0$, $a_1 = 3$ and

$$a_n = 4a_{n-1} - 4a_{n-2} \text{ for } n > 1.$$

Solution:
The characteristic polynomial is $x^2 - 4x + 4 = (x-2)^2$ with the only root $x_0 = 2$. The solution is therefore of form $a_n = \lambda 2^n + \mu n 2^n$. Our constraints yield $\lambda = 0$ and $\mu = \frac{3}{2}$.

Exercise H5
Use the repertoire method to find a closed form for the following recurrence:

$$\begin{aligned} a_0 &= 5 \\ a_1 &= 9 \\ a_n &= na_{n-1} + n^2 a_{n-2} - n^4 - 3n^2 + 5 \qquad \text{for } n \geq 2 \end{aligned}$$

Solution:
Let $f(n) = -n^4 - 3n^2 + 5$, i.e., $f(n) = a_n - na_{n-1} - n^2 a_{n-2}$.

a_n	$f(n)$	a_0	a_1
1	$-n^2 - n + 1$	1	1
n	$-n^3 + n^2 + 2n$	0	1
n^2	$-n^4 + 3n^3 - n^2 - n$	0	1

Let Z_i for $i = 1, 2, 3$ be the solutions of the first, second, and third line, respectively. Then $f(n) = 5Z_1 + 3Z_2 + Z_3$. For these, a_0 and a_1 are correct, and thus $a_n = 5 \cdot 1 + 3n + n^2$.

Exercise H6
Solve the following recurrence and find a nice representation of the solution (in a mathematical sense).

$$\begin{aligned} c_0 &= 2 \\ c_1 &= 4 \\ c_n &= c_{n-2}^{\log c_{n-1}} \end{aligned}$$

Hint: Let F_n be the nth Fibonacci number. Write c_n as some function of F_n.

Solution:
We apply the logarithm and obtain

$$\log c_n = \log c_{n-1} \cdot \log c_{n-2}.$$

In order to obtain a sum instead of a product, we repeat this and obtain

$$\log\log c_n = \log\log c_{n-1} + \log\log c_{n-2}.$$

Substituting $d_n = \log\log c_n$ yields

$$\begin{aligned} d_0 &= 0 \\ d_1 &= 1 \\ d_n &= d_{n-1} + d_{n-2} \end{aligned}$$

Since this describes the Fibonacci numbers, we obtain $c_n = 2^{2^{F_n}}$, where F_n denotes the n-th Fibonacci number.