Analysis of Algorithms

Recall from class that if a complex-valued function $f(z)$ is analytic at 0 and R is the absolute value of the singularity nearest to the origin then the coefficient $f_n = [z^n]f(z)$ satisfies:

$$f_n \asymp \left(\frac{1}{R}\right)^n.$$

Exercise 11-1
Sort the following generating functions *within one minute* by their exponential growth!

1. $A(z) = 1/\sqrt{1 - z/2}$
2. $B(z) = (1 + z)/(1 - z)$
3. $C(z) = \frac{1}{1-e^{-z-1/3}}$

Solution: We only need to sort them by the absolute value of the dominant singularities. $A_n \asymp 1/2^n$, $B_n \asymp 1$, $C_n \asymp 3^n$.

Exercise 11-2
An algorithm is given an array of length $n \geq 0$ and, if $n \geq 2$, for each $1 \leq k \leq n$ calls itself on some random subarray of length k with probability $\frac{1}{2}$. Compute the exponential growth of the running time of this algorithm.
Solution:
Let t_n be the expected number of calls of the algorithm for an array of length n. Then $t_0 = t_1 = 1$ and $t_n = 1 + \frac{1}{2}\sum_{k=1}^{n} t_k$ for $n > 1$. To get an equation that holds for all n, we let

$$t_n = 1 + \frac{1}{2}\sum_{k=1}^{n} t_k - \frac{1}{2}(n = 1).$$

We make yet another modification so that in the equation above so that we may sum from $k = 0$:

$$t_n = 1 + \frac{1}{2}\sum_{k=0}^{n} t_k - \frac{1}{2}(n = 1) - \frac{1}{2}.$$

Define $T(z) = \sum_{n=0}^{\infty} t_n z^n$. Multiplying both sides by z^n and summing over n, we obtain:

$$T(z) = \frac{1}{1-z} + \frac{1}{2(1-z)}T(z) - \frac{1}{2}z + \frac{1}{2(1-z)}$$

which gives the following functional relation: $T(z) = \frac{z^2-z+3}{1-2z}$. Since the dominant singularity is located at $z = \frac{1}{2}$, we get an asympotic running time of $\asymp 2^n$.

Exercise 11-3
Determine $[z^n]\frac{1}{2-e^z}$ up to an additive error of $O(12^{-n})$!
Solution: We first determine the dominant singularity, which is $\ln 2$. Since, by L'Hôpital,

$$\lim_{z\to\ln 2}\frac{\ln 2 - z}{2-e^z}=\frac{1}{2},$$

we have

$$\frac{\ln 2 - z}{\ln 2 - z}\cdot\frac{1}{2-e^z}\sim\frac{1}{2}\cdot\frac{1}{\ln 2 - z}=\frac{1}{2\ln 2}\cdot\frac{1}{1-(1/\ln 2)z}=\frac{1}{2\ln 2}\sum_{n=0}^{\infty}\left(\frac{1}{\ln 2}\right)^n z^n.$$

We now look for the next singularites (ordered by their distance from the origin). These are $\ln 2 \pm 2\pi i$. By L'Hôpital

$$\lim_{z\to\ln 2\pm 2\pi i}\frac{z-\ln 2\mp 2\pi i}{2-e^z}=-\frac{1}{2}$$

und therefore

$$S(z)\sim -\frac{1}{2}\frac{1}{z-\ln 2\mp 2\pi i}=\frac{1}{2}\frac{1}{\ln 2\mp 2\pi i}\frac{1}{1-\frac{z}{\ln 2\mp 2\pi i}}$$

for $z\to\ln 2\pm 2\pi i$. We can now use the Theorem of the lecture to determine the coefficients:

$$\begin{aligned}[z^n]S(z) &= \frac{1}{2}\left((\frac{1}{\ln 2})^{n+1}+(\frac{1}{\ln 2+2\pi i})^{n+1}+(\frac{1}{\ln 2-2\pi i})^{n+1}\right)+O(r)^{-n}\\ &= \frac{1}{2}\left((\frac{1}{\ln 2})^{n+1}+r^{n+1}(e^{i\phi(n+1)}+e^{-i\phi(n+1)})\right)+O(r)^{-n}\\ &= \frac{1}{2}\left((\frac{1}{\ln 2})^{n+1}+r^{n+1}\cos(\phi(n+1))\right)+O(r)^{-n}\end{aligned}$$

with $r=1/\sqrt{\ln^2 2+4\pi^2}\approx 12.58547409739904$, $\phi=\arctan(\frac{2\pi}{\ln 2})$.

Homework Assignment 11-1 (10 points)
Determine the exponential growth of $[z^n]G(z)$, where

a) $G(z)=z^2/(1-z-z^2)$,

b) $G(z)=\sqrt{1+2z}-\sqrt{2+2z-4z^2}/\sqrt{3}$,

c) $G(z)=\ln(1+\sin(z))/\ln(1+\cos(z))$.

Homework Assignment 11-2 (10 points)

$$A(z)=\frac{\sqrt{1-z^7}}{2z^2-3z+1}\qquad B(z)=\frac{1-z^2}{e^{z+3z^2}}\qquad C(z)=z^5+3z^2(z^3+z^2+8)$$

Sort the sequences a_n, b_n, and c_n by their exponential growth.

Homework Assignment 11-3 (10 Points)
Determine $g_n=[z^n]G(z)$ up to an additive error of $O(4^n)$, where

$$G(z)=\sum_{n=0}^{\infty}g_n z^n=\frac{15z^2+8z+1}{15z^2-8z+1}.$$