

## Analysis of Algorithms

This tutorial is geared towards solving recurrence relations using generating functions. Here is an observation: If  $a_n$  satisfies the recurrence

$$a_n = x_1 a_{n-1} + x_2 a_{n-2} + \cdots + x_t a_{n-t}$$

for  $n \geq t$ , then the generating function  $a(z) = \sum_{n \geq 0} a_n z^n$  is a rational function  $a(z) = f(z)/g(z)$ , where the denominator polynomial  $g(z) = 1 - x_1 z - x_2 z^2 - \cdots - x_t z^t$  and the numerator polynomial is determined by the initial values  $a_0, a_1, \dots, a_{t-1}$  and has degree at most  $t - 1$ . In fact,  $f(z)$  may be written as:

$$f(z) = g(z) \sum_{n=0}^{t-1} a_n z^n \pmod{z^t}.$$

### Problem 9-1

Solve the following recurrence relations:

1.  $a_n = 2a_{n-1} + a_{n-2} - 2a_{n-3}$  for  $n > 2$  with  $a_0 = 0$  and  $a_1 = a_2 = 1$ .
2.  $a_n = 5a_{n-1} - 8a_{n-2} + 4a_{n-3}$  for  $n > 2$  with  $a_0 = 0$ ,  $a_1 = 1$ , and  $a_2 = 4$ .

### Solution

1. First write down  $g(z)$  as  $1 - 2z - z^2 + 2z^3$ . Using the initial conditions,  $f(z)$  may be written as:

$$\begin{aligned} f(z) &= g(z) \sum_{n=0}^2 a_n z^n \pmod{z^3} \\ &= (z + z^2)(1 - 2z - z^2 + 2z^3) \pmod{z^3} \\ &= z - z^2 = z(1 - z). \end{aligned}$$

Therefore,

$$a(z) = \frac{z(1 - z)}{1 - 2z - z^2 + 2z^3} = \frac{z(1 - z)}{(1 - z)(1 + z)(1 - 2z)} = \frac{z}{(1 + z)(1 - 2z)}.$$

Using partial fractions, the last expression on the right hand side works out to:

$$\frac{z}{(1 + z)(1 - 2z)} = \frac{1}{3} \left( \frac{1}{1 - 2z} - \frac{1}{1 + z} \right),$$

from which we can read off  $a_n = \frac{1}{3}(2^n - (-1)^n)$ .

2. We start by writing down  $g(z)$  as  $1 - 5z + 8z^2 - 4z^3$ . Now using the initial conditions,  $f(z)$  can be written as:

$$\begin{aligned} f(z) &= g(z) \sum_{n=0}^2 a_n z^n = g(z)(z + 4z^2) \pmod{z^3} \\ &= (1 - 5z + 8z^2 + 4z^3)(z + 4z^2) \pmod{z^3} \\ &= z(1 - z). \end{aligned}$$

Now  $1 - 5z + 8z^2 - 4z^3$  can be factorized as  $(1 - z)(1 - 2z)^2$ , so that  $a(z) = z/(1 - 2z)^2$ . We know that

$$\frac{2z}{(1 - 2z)^2} = \sum_{n=0}^{\infty} n(2z)^n,$$

and so  $a_n = n2^{n-1}$  for  $n \geq 3$ . Actually, substituting  $n = 0, 1, 2$ , we see that this holds for  $n \geq 0$ .

### Problem 9-2

In this problem, we will solve the *Quicksort* recurrence using OGFs. Typically when the coefficients of the recurrence are polynomials in the index  $n$ , then the relationship constraining the generating function is a differential equation. Recall that the Quicksort recurrence is:

$$nC_n = n(n + 1) + 2 \sum_{k=1}^n C_{k-1}, \text{ for } n \geq 1 \text{ with } C_0 = 0.$$

Define  $C(z) = \sum_{n \geq 0} C_n z^n$ .

1. Multiply both sides of the recurrence by  $z^n$  and sum over the index  $n$  to obtain a functional relation involving  $C(z)$ ,  $C'(z)$ , and  $z$  of the form:

$$C'(z) + P(z)C(z) = Q(z),$$

where  $P(z)$  and  $Q(z)$  are functions of  $z$ .

2. This differential equation can be solved by multiplying both sides by the “integrating factor”  $e^{\int_0^z P(x) dx}$ . Notice that multiplying by this factor yields:

$$\begin{aligned} C'(z)e^{\int_0^z P(x) dx} + P(z)C(z)e^{\int_0^z P(x) dx} &= Q(z)e^{\int_0^z P(x) dx} \\ \left( C(z)e^{\int_0^z P(x) dx} \right)' &= Q(z)e^{\int_0^z P(x) dx} \end{aligned}$$

We can now express  $C(z)$  as

$$C(z) = e^{-\int_0^z P(x) dx} \int Q(z)e^{\int_0^z P(x) dx} dz.$$

## Solution

As suggested, multiplying both sides of the recurrence by  $z^n$  and then summing over the index  $n$  yields

$$\sum_{n=0}^{\infty} nC_n z^n = \sum_{n=0}^{\infty} n(n+1)z^n + 2 \sum_{n=0}^{\infty} \left( \sum_{k=1}^n C_{k-1} \right) z^n. \quad (1)$$

Now  $\sum_{n \geq 0} nC_n z^n = zC'(z)$ . Note that

$$\begin{aligned} \sum_{n=0}^{\infty} n(n+1)z^n &= 2 \sum_{n=0}^{\infty} \binom{n+1}{2} z^n \\ &= \frac{2}{z} \sum_{n=0}^{\infty} \binom{n+1}{2} z^{n+1} \\ &= \frac{2}{z} \sum_{n=2}^{\infty} \binom{n}{2} z^n \\ &= \frac{2}{z} \frac{z^2}{(1-z)^3} \\ &= \frac{2z}{(1-z)^3}. \end{aligned}$$

Finally, note that

$$\begin{aligned} \sum_{n=0}^{\infty} \left( \sum_{k=1}^n C_{k-1} \right) z^n &= \sum_{n=0}^{\infty} z^n \sum_{n=0}^{\infty} C_n z^{n+1} \\ &= \frac{zC(z)}{1-z}. \end{aligned}$$

Hence the functional relation that we are seeking is:

$$C'(z) - \frac{2}{1-z}C(z) = \frac{2}{(1-z)^3}.$$

The integrating factor in our case is

$$e^{-2 \int_{x=0}^z \frac{1}{1-z} dx} = e^{2 \ln(1-z)} = (1-z)^2.$$

Hence  $C(z)$  is given by

$$C(z) = \frac{1}{(1-z)^2} \int \frac{2}{1-z} dz = \frac{2}{(1-z)^2} \ln \frac{1}{1-z}.$$

Using the fact that  $\frac{z}{(1-z)^2} \ln \frac{1}{1-z} = \sum_{n \geq 0} n(H_n - 1)z^n$ , we obtain:

$$[z^n] \frac{2}{(1-z)^2} \ln \frac{1}{1-z} = 2(n+1)(H_{n+1} - 1).$$

### Homework Assignment 9-1 (10 Points)

Solve the recurrences:

1.  $a_n = 5a_{n-1} - 8a_{n-2} + 4a_{n-3}$  for  $n > 2$  with  $a_0 = 1$ ,  $a_1 = 2$ , and  $a_2 = 4$ .

2.  $a_n = 3a_{n-1} - 3a_{n-2} + a_{n-3}$  for  $n > 2$  with  $a_0 = a_1 = 0$  and  $a_2 = 1$ .

*Hint:* Use might need to use:

$$\frac{1}{(1-z)^{m+1}} = \sum_{n=0}^{\infty} \binom{n+m}{m} z^n.$$

### Homework Assignment 9-2 (10 points)

Use generating functions to solve the recurrence:

$$na_n = (n-2)a_{n-1} + 2 \quad \text{for } n > 2 \quad \text{with } a_1 = 1, a_2 = 1.$$