## Analysis of Algorithms

### Problem 8-1

Find the ordinary generating function (OGF) for each of the following sequences:

(1) $\{k2^{k+1}\}_{k\geq 0}$ (2) $s_0 = 0$ and $s_k = \frac{1}{k}$ for $k \geq 1$ (3) $\{H_k\}_{k\geq 1}$
(4) $\{kH_k\}_{k\geq 1}$ (5) $\{k^3\}_{k\geq 2}$

### Solution

1. The ordinary generating function for the sequence $\{k2^{k+1}\}_{k\geq 0}$ is:

$$\sum_{k\geq 0} k2^{k+1}z^k = \sum_{k\geq 0} 2k(2z)^k = 2z\sum_{k\geq 1} 2k(2z)^{k-1} = 2z\left(\sum_{k\geq 0}(2z)^k\right)'.$$

   Now the last term may be rewritten as:

$$2z\left(\frac{1}{1-2z}\right)' = \frac{4z}{(1-2z)^2}.$$

2. Start with the generating function $G(z) = \sum_{k=0}^{\infty} z^k = \frac{1}{1-z}$. Integrating both sides we obtain:

$$\begin{aligned}\int_0^z \sum_{k=0}^{\infty} z^k \mathrm{d}z &= \int_0^z \frac{1}{1-z}\mathrm{d}z \\ &= -\ln(1-z)\,\big|_0^z \\ &= \ln\frac{1}{1-z}.\end{aligned}$$

   The series itself can be integrated term-by-term. This is valid if $z$ is in the radius of convergence of the power series, and we can make this assumption because the value of $z$ itself is not of interest to us. This then gives us:

$$\ln\frac{1}{1-z} = \sum_{k=0}^{\infty}\frac{z^{k+1}}{k+1} = \sum_{k=1}^{\infty}\frac{z^k}{k}.$$

   Since $s_0 = 0$, the OGF of the sequence is $\ln(1-z)^{-1}$.

3. Let $s_k$ be as defined before: $s_0 = 0$ and $s_k = 1/k$ for $k \geq 1$. Also define $H_0 = 0$. Then the OGF for $\{H_k\}_{k\geq 0}$ is

$$\begin{aligned}
\sum_{k=0}^{\infty} H_k z^k &= \sum_{k=0}^{\infty} \left( H_0 + 1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{k} \right) z^k \\
&= \sum_{k=0}^{\infty} \left( \sum_{i=0}^{k} 1 \cdot s_{k-i} \right) z^k \\
&= \sum_{k=0}^{\infty} z^k \sum_{k=0}^{\infty} s_k z^k \\
&= \frac{1}{1-z} \ln \frac{1}{1-z}.
\end{aligned}$$

4. We start with the identity obtained in the last exercise: $\sum_{k\geq 0} H_k z^k = \frac{1}{1-z} \ln \frac{1}{1-z}$. Differentiating both sides and multiplying by $z$, we obtain:

$$\sum_{k=1}^{\infty} k H_k z^k = \frac{z}{(1-z)^2} \left( 1 + \ln \frac{1}{1-z} \right).$$

Now the index $k$ in power series on the left hand side may be written so that it ranges from 0 to $\infty$. This shows that the expression on the left hand side is the closed-form of the generating function that we are seeking.

5. We start with the fact that $\sum_{k\geq 0} z^k = \frac{1}{1-z}$. Differentiating both sides and using the fact that a power series can be termwise differentiated w.r.t $z$ if $z$ is in the radius of convergence, we obtain:

$$\sum_{k=1}^{\infty} k z^{k-1} = \ln \frac{1}{1-z}. \tag{1}$$

Multiplying both sides by $z$ and differentiating again, we obtain:

$$\sum_{k=1}^{\infty} k^2 z^{k-1} = \ln \frac{1}{1-z} + \frac{z}{1-z}. \tag{2}$$

We again multiply both sides by $z$ and differentiate, to obtain:

$$\sum_{k=1}^{\infty} k^3 z^{k-1} = \ln \frac{1}{1-z} + \frac{3z}{1-z} + \frac{z^2}{(1-z)^2}. \tag{3}$$

Again multiplying the above by $z$, we obtain:

$$\sum_{k=1}^{\infty} k^3 z^k = z \ln \frac{1}{1-z} + \frac{3z^2}{1-z} + \frac{z^3}{(1-z)^2}. \tag{4}$$

Now the generating function for our sequence is $\sum_{k\geq 2} k^3 z^k$ and this equals the right hand side of (4) minus $z$, that is:

$$z \ln \frac{1}{1-z} + \frac{3z^2}{1-z} + \frac{z^3}{(1-z)^2} - z.$$

**Problem 8-2**

```
1 func( int n ){
    int s = 0;
3   if ( n == 0 ) return 1;
    for( int i = 0; i < n; ++i )
5     s += func( i );
    return s;
7 }
```

Compute how often the 5th line of this program is executed using (ordinary) generating functions.

**Solution**

Let $A_n$ denote the how often the 5th line is called on input $n$. We immediately obtain $A_0 = 0$ and $A_n = n + \sum_{i=0}^{n-1} A_i$. The corresponding generating function is thus

$$A(z) := \sum_{n=0}^{\infty} A_n z^n = \sum_{n=0}^{\infty} n z^n + \sum_{n=0}^{\infty} \left( \sum_{k=0}^{n-1} A_k \right) z^n.$$

Now the last term on the right hand side may be seen as the convolution of the series $\sum_{n=0}^{\infty} z^n$ and $\sum_{n=0}^{\infty} A_n z^{n+1}$. The convolution product is simply $zA(z)/(1-z)$ so that we may write:

$$A(z) = \frac{z}{(1-z)^2} + \frac{zA(z)}{1-z},$$

which implies:

$$A(z)(1-2z) = \frac{z}{1-z}$$

and therefore

$$A(z) = \frac{1}{(1-z)(1-2z)} z.$$

Let $\frac{1}{(1-z)(1-2z)} = \frac{a}{1-z} + \frac{b}{1-2z}$ for some $a, b$. Setting $z = 0$ implies $a + b = 1$ and $z = -1$ implies $a/2 + b/3 = 1/6$. We easily obtain $a = -1$ and $b = 2$ and therefore

$$A(z) = \left( -\frac{1}{1-z} + \frac{2}{1-2z} \right) z = -\sum_{n=0}^{\infty} z^{n+1} + \sum_{n=0}^{\infty} (2z)^{n+1} = \sum_{n=1}^{\infty} (2^n - 1) z^n.$$

Thus $A_0 = 0$ and $A_n = 2^n - 1$ for all $n \geq 1$.

**Homework Assignment 8-1 (10 Points)**

Solve this recurrence using generating functions:

$$a_n = 2a_{n-1} + 3a_{n-2}$$

and $a_0 = 0$, $a_1 = 2$.

**Homework Assignment 8-2 (10 points)**

Find $[z^n]$ for each of the following OGFs.

$$\frac{1}{(1-3z)^4}, \quad (1-z)^2 \ln \frac{1}{1-z}, \quad \frac{1}{(1-2z^2)^2}$$