Analysis of Algorithms

This tutorial is geared towards using generating functions for various counting problems.

Problem 10-1

Here is a classical problem: n gentlemen attend a party and check their hats. The checker has a little too much drink and returns the hats at random. What is the probability that no gentlemen receives his own hat? How does the probability depend on the number of gentlemen?

Solution

Let the n gentlemen be labeled $1, 2, \ldots, n$. A permutation of $\{1, \ldots, n\}$ in which element i is not placed at position i, for any i, is called a *derangement*. For example, for $n = 3$, 312 is a derangement but 321 is not as 2 is in the second place.

Let D_n denote the number of derangements of n elements. Clearly $D_1 = 0$. $D_2 = 1$ as 21 is the only derangement. We will define $D_0 = 1$. It is convenient to say that there is one permutation of the empty set and that it does not map anything to itself.

Consider the general case with $n+1$ elements. Element 1 has to be at some position k, where $2 \leq k \leq n+1$. Now there are two possibilities. Either element k is at position 1, in which case there are D_{n-1} derangements possible. Otherwise, some other element is at position 1. This second situation may also be viewed as follows: We keep element 1 fixed at the first position; derange elements $2, \ldots, n+1$ in D_n ways; finally, exchange the elements at the first and kth positions to obtain a derangement of the elements $1, \ldots, n+1$. The recurrence for D_{n+1} may now be written as:

$$D_{n+1} = n(D_n + D_{n-1}). \tag{1}$$

Using the above recurrence, we can write $D_{n+1} - (n+1)D_n$ as:

$$\begin{aligned}
D_{n+1} - (n+1)D_n &= nD_{n-1} - D_n \\
&= -\left(D_n - nD_{n-1}\right) \\
&= (-1)^2\left(D_{n-1} - (n-1)D_{n-2}\right) \\
&= (-1)^3\left(D_{n-2} - (n-2)D_{n-3}\right) \\
&\ \ \vdots \\
&= (-1)^{n-1}(D_2 - 2D_1).
\end{aligned}$$

Put differently, the recurrence (1) may be expressed as:

$$D_{n+1} = (n+1)D_n + (-1)^{n+1} \quad \text{where} \quad n \geq 2. \tag{2}$$

Define $D(z) = \sum_{n=0}^{\infty} D_n \frac{z^n}{n!}$. Multiply both sides by $z^{n+1}/(n+1)!$ and sum over n, obtaining:

$$\sum_{n=0}^{\infty} D_{n+1}\frac{z^{n+1}}{(n+1)!} = \sum_{n=0}^{\infty}(n+1)D_n\frac{z^{n+1}}{(n+1)!} + \sum_{n=0}^{\infty}(-1)^{n+1}\frac{z^{n+1}}{(n+1)!}$$

The left-hand-side is $D(z) - D_0$. The first term on the right-hand-side is $zD(z)$ and the second term is $e^{-z} - 1$. Thus the above equation may be written as:

$$D(z) = \frac{e^{-z}}{1-z} = \sum_{n=0}^{\infty}(-1)^n \frac{z^n}{n!} \sum_{n=0}^{\infty} z^n,$$

from which we may write down D_n as:

$$D_n = n! \left(1 - \frac{1}{1!} + \frac{1}{2!} - \frac{1}{3!} + \cdots + (-1)^n \frac{1}{n!}\right).$$

Thus the probability that no gentleman recieves his own hat is $D_n/n!$ which approaches $e^{-1} = 0.3678\ldots$ This is independent of n.

Problem 10-2

Suppose we are given a sequence of n terms $x_1, x_2, \ldots, x_n$. We are interested in finding out the number of ways of parenthesizing this sequence. We assume that it is forbidden to insert parenthesis around a single term, like this: (x_1). We therefore have to find out the number of ways of inserting $n-1$ left parentheses and $n-1$ right parentheses into the sequence $x_1, \ldots, x_n$ such that as we go from left to right, the number of right parentheses never exceeds the number of left parentheses. For $n = 1, 2, 3, 4$, we list all valid parenthesizations below:

P_1	P_2	P_3	P_4
x_1	(x_1x_2)	$((x_1x_2)x_3)$	$(((x_1x_2)x_3)x_4)$
		$(x_1(x_2x_3))$	$(x_1(x_2(x_3x_4)))$
			$((x_1(x_2x_3))x_4)$
			$(x_1((x_2x_3)x_4))$
			$((x_1x_2)(x_3x_4))$

In how many ways can we parenthesize an expression with n terms?

Solution

Let P_n be the number of ways to parenthesize a sequence of n terms such as the one given. For $n \geq 2$, there are essentially two possible ways of parenthesizing. The first one looks like this:

$$((x_1 \ldots x_r)(x_{r+1} \ldots x_n)),$$

where $1 < r < n-1$. The second one accounts for the two cases: $r = 1$ and $r = n-1$, and looks like:

$$(x_1(x_2 \ldots x_n)) \quad \text{or} \quad ((x_1 \ldots x_{n-1})x_n)$$

In either case, there are P_r ways of parenthesizing the first r terms and P_{n-r} ways of parenthesizing the last $n-r$ terms. Therefore we obtain:

$$P_n = \sum_{r=1}^{n-1} P_r P_{n-r}. \tag{3}$$

Define $P_0 = 0$ so that we can write $P_n = \sum_{r=0}^{n} P_r P_{n-r}$; also define $P(z) = \sum_{n\geq 0} P_n z^n$.

Note that we cannot conclude $P(z) = P(z)^2$, since recurrence (3) holds only for $n \geq 2$. To get around this, define a new sequence Q_n such that:

$$Q_n = \begin{cases} P_0P_0 = 0 & \text{if } n = 0 \\ P_0P_1 + P_1P_0 = 0 & \text{if } n = 1 \\ P_n & \text{if } n \geq 2 \end{cases}$$

and the OGF $Q(z) = \sum_{n\geq 0} Q_n z^n$. Then $Q(z) = P(z)^2$ and $Q(z) = P(z) - z$ resulting in the functional equation:

$$P(z)^2 - P(z) + z = 0,$$

which yields:

$$P(z) = \frac{1 \pm \sqrt{1-4z}}{2}.$$

Since $P(0) = 0$, we choose the negative sign. Expanding $P(z)$ using the binomial theorem, we obtain that:

$$P_n = \frac{1}{n}\binom{2n-2}{n-1}.$$

Homework Assignment 10-1 (10 Points)

1. Find the EGFs for $1, 3, 5, 7, \ldots$ and $0, 2, 4, 6, \ldots$
2. Find the coefficient of $z^n/n!$ for each of the following EGFs

$$A(z) = \frac{1}{1-z}\ln\frac{1}{1-z}, \quad A(z) = e^{z+z^2}.$$

Solution

The EGF for the first sequence is:

$$1 \cdot \frac{z^0}{0!} + 3 \cdot \frac{z^1}{1!} + 5 \cdot \frac{z^2}{2!} + \cdots = \sum_{n=0}^{\infty}(2n+1)\frac{z^n}{n!}.$$

We may write this sum as:

$$\begin{aligned} \sum_{n=0}^{\infty}(2n+1)\frac{z^n}{n!} &= 2z\sum_{n=1}^{\infty}\frac{z^{n-1}}{(n-1)!} + \sum_{n=0}^{\infty}\frac{z^n}{n!} \\ &= 2ze^z + e^z. \end{aligned}$$

The EGF of the second sequence is:

$$0 \cdot \frac{z^0}{0!} + 2 \cdot \frac{z^1}{1!} + 4 \cdot \frac{z^2}{2!} + \cdots = \sum_{n=0}^{\infty} 2n\frac{z^n}{n!}.$$

This is $2ze^z$ as shown above.

Now we know that the function $\frac{1}{1-z}\ln\frac{1}{1-z}$ is the OGF of the sequence H_n. That is,

$$\sum_{n=1}^{\infty} H_n z^n = \frac{1}{1-z}\ln\frac{1}{1-z}.$$

Thus the coefficient of $n![z^n]$ when this function is regarded as an EGF is simply $n!H_n$. The function $e^{z+z^2} = e^z e^{z^2}$ and this may be written as:

$$\begin{aligned} e^z e^{z^2} &= \sum_{n=0}^{\infty} \frac{z^n}{n!} \cdot \sum_{n=0}^{\infty} \frac{z^{2n}}{n!} \\ &= \sum_{n=0}^{\infty} \frac{z^n}{n!} \cdot \sum_{n=0}^{\infty} (n \text{ even}) \cdot \frac{z^n}{(n/2)!} \\ &= \sum_{n=0}^{\infty} \frac{z^n}{n!} \cdot \sum_{n=0}^{\infty} (n \text{ even}) \cdot \frac{n!}{(n/2)!} \cdot \frac{z^n}{n!}. \end{aligned}$$

The coefficent of $n![z^n]$ is therefore:

$$n! \sum_{k=0}^{n} \frac{1}{(n-k)!} \cdot (k \text{ even}) \cdot \frac{k!}{(k/2)!}.$$

We may rewrite this as follows:

$$n! \sum_{r=0}^{\lfloor n/2 \rfloor} \frac{1}{(n-2r)!} \cdot \frac{(2r)!}{r!},$$

which is the coefficient of $n![z^n]$.

Homework Assignment 10-2 (10 points)

Call a sequence of **push** und **pop** operations ($\uparrow$ and $\downarrow$) *valid*, if it contains the same number of $\uparrow$ and $\downarrow$ and no prefix of the sequence consists of fewer $\uparrow$ than $\downarrow$. For example, $(\uparrow, \uparrow, \downarrow, \downarrow, \uparrow, \downarrow)$ is valid, while $(\downarrow, \downarrow, \uparrow, \uparrow)$ and $(\uparrow, \downarrow, \downarrow, \uparrow)$ are not valid. The number of $\uparrow$s in a valid sequence is called the *length* of the sequence. How many valid sequences of length n are there?

Solution

Let P_n be the number of valid sequences of length n, where the length of a sequence is redefined to be the total number of pushes and pops. In particular, lengths are always even. If we can solve this, then the number of n-length sequences (where the length is as defined in the problem) is precisely P_{2n}. Thus for our redefinition of length, we have $P_0 = 1$, $P_1 = 0$, $P_2 = 1$ and $P_{2k+1} = 0$ for all k.

Let n be even. Any valid sequence with $n/2$ pushes and $n/2$ pops looks like:

$$\uparrow P_k \downarrow P_{n-2-k},$$

which allows us to write: $P_n = \sum_{k=0}^{n-2} P_k P_{n-2-k}$, where $n \geq 2$. To see what this convolution looks like, we write down a few initial values:

$$\begin{aligned} P_2 &= P_0 P_0 = 1 \\ P_3 &= P_0 P_1 + P_1 P_0 = 1 \\ P_4 &= P_0 P_2 + P_2 P_0 = 2 \\ P_5 &= P_0 P_3 + P_1 P_2 + P_2 P_1 + P_3 P_0 = 0 \end{aligned}$$

Define $P(z) = \sum_{n=0}^{\infty} P_n z^n$ and note that we cannot conclude that $P(z) = P(z)^2$. Define a new sequence Q_n such that

$$Q(z) = \sum_{n=0}^{\infty} Q_n z^n = P(z)^2,$$

as follows: $Q_0 = P_2$, $Q_1 = P_3$ and, in general, $Q_n = P_{n+2}$. With this definition, it follows that $Q(z) = P(z)^2$. We may write $Q(z)$ differently as follows:

$$Q(z) = \frac{P(z) - P_0 - P_1 z}{z^2} = \frac{P(z) - 1}{z^2},$$

which yields the following functional relationship: $P(z)^2 z^2 - P(z) + 1 = 0$. Thus

$$P(z) = \frac{1 \pm \sqrt{1 - 4z^2}}{2z^2},$$

where we take the minus sign because then $P(0) = P_0 = 1$, which is what it should be. (Verify this!).

We can now use the Binomial Theorem and write $P(z)$ as:

$$\begin{aligned} P(z) &= \frac{1 - \sqrt{1 - 4z^2}}{2z^2} \\ &= \frac{1}{2z^2} \left(1 - \sum_{n=0}^{\infty} \binom{1/2}{n} (-4)^n z^{2n} \right) \\ &= \frac{1}{2z^2} \left(- \sum_{n=1}^{\infty} \binom{1/2}{n} (-4)^n z^{2n} \right) \\ &= - \sum_{n=1}^{\infty} \binom{1/2}{n} (-1)^n 2^{2n-1} z^{2n-2}. \end{aligned}$$

Observe that when $n = 1$, then the exponent of z is 0 and the corresponding coefficient is P_0. Verify that for any $r \geq 0$, if $n = r + 1$ the exponent is $2r$ and the coefficient is P_{2r}. The coefficient that we are interested in is

$$P_{2n} = - \binom{1/2}{n+1} (-1)^{n+1} 2^{2(n+1)-1}.$$

This may be simplified as follows:

$$\begin{aligned} P_{2n} &= \binom{1/2}{n+1} (-1)^{n+2} 2^{2n+1} \\ &= \frac{(-1)^{n+2} 2^{2n+1}}{(n+1)!} \cdot \frac{1}{2} \cdot \left(\frac{1}{2} - 1 \right) \cdot \left(\frac{1}{2} - 2 \right) \cdot \left(\frac{1}{2} - n \right) \\ &= \frac{1}{n+1} \binom{2n}{n}. \end{aligned}$$

Verify the calculation!