

Analysis of Algorithms

This tutorial is geared towards solving recurrence relations using generating functions. Here is an observation: If a_n satisfies the recurrence

$$a_n = x_1 a_{n-1} + x_2 a_{n-2} + \cdots + x_t a_{n-t}$$

for $n \geq t$, then the generating function $a(z) = \sum_{n \geq 0} a_n z^n$ is a rational function $a(z) = f(z)/g(z)$, where the denominator polynomial $g(z) = 1 - x_1 z - x_2 z^2 - \cdots - x_t z^t$ and the numerator polynomial is determined by the initial values a_0, a_1, \dots, a_{t-1} and has degree at most $t - 1$. In fact, $f(z)$ may be written as:

$$f(z) = g(z) \sum_{n=0}^{t-1} a_n z^n \pmod{z^t}.$$

Problem 9-1

Solve the following recurrence relations:

1. $a_n = 2a_{n-1} + a_{n-2} - 2a_{n-3}$ for $n > 2$ with $a_0 = 0$ and $a_1 = a_2 = 1$.
2. $a_n = 5a_{n-1} - 8a_{n-2} + 4a_{n-3}$ for $n > 2$ with $a_0 = 0$, $a_1 = 1$, and $a_2 = 4$.

Solution

1. First write down $g(z)$ as $1 - 2z - z^2 + 2z^3$. Using the initial conditions, $f(z)$ may be written as:

$$\begin{aligned} f(z) &= g(z) \sum_{n=0}^2 a_n z^n \pmod{z^3} \\ &= (z + z^2)(1 - 2z - z^2 + 2z^3) \pmod{z^3} \\ &= z - z^2 = z(1 - z). \end{aligned}$$

Therefore,

$$a(z) = \frac{z(1 - z)}{1 - 2z - z^2 + 2z^3} = \frac{z(1 - z)}{(1 - z)(1 + z)(1 - 2z)} = \frac{z}{(1 + z)(1 - 2z)}.$$

Using partial fractions, the last expression on the right hand side works out to:

$$\frac{z}{(1 + z)(1 - 2z)} = \frac{1}{3} \left(\frac{1}{1 - 2z} - \frac{1}{1 + z} \right),$$

from which we can read off $a_n = \frac{1}{3}(2^n - (-1)^n)$.

2. We start by writing down $g(z)$ as $1 - 5z + 8z^2 - 4z^3$. Now using the initial conditions, $f(z)$ can be written as:

$$\begin{aligned} f(z) &= g(z) \sum_{n=0}^2 a_n z^n = g(z)(z + 4z^2) \pmod{z^3} \\ &= (1 - 5z + 8z^2 + 4z^3)(z + 4z^2) \pmod{z^3} \\ &= z(1 - z). \end{aligned}$$

Now $1 - 5z + 8z^2 - 4z^3$ can be factorized as $(1 - z)(1 - 2z)^2$, so that $a(z) = z/(1 - 2z)^2$. We know that

$$\frac{2z}{(1 - 2z)^2} = \sum_{n=0}^{\infty} n(2z)^n,$$

and so $a_n = n2^{n-1}$ for $n \geq 3$. Actually, substituting $n = 0, 1, 2$, we see that this holds for $n \geq 0$.

Problem 9-2

In this problem, we will solve the *Quicksort* recurrence using OGFs. Typically when the coefficients of the recurrence are polynomials in the index n , then the relationship constraining the generating function is a differential equation. Recall that the Quicksort recurrence is:

$$nC_n = n(n + 1) + 2 \sum_{k=1}^n C_{k-1}, \text{ for } n \geq 1 \text{ with } C_0 = 0.$$

Define $C(z) = \sum_{n \geq 0} C_n z^n$.

1. Multiply both sides of the recurrence by z^n and sum over the index n to obtain a functional relation involving $C(z)$, $C'(z)$, and z of the form:

$$C'(z) + P(z)C(z) = Q(z),$$

where $P(z)$ and $Q(z)$ are functions of z .

2. This differential equation can be solved by multiplying both sides by the “integrating factor” $e^{\int_0^z P(x) dx}$. Notice that multiplying by this factor yields:

$$\begin{aligned} C'(z)e^{\int_0^z P(x) dx} + P(z)C(z)e^{\int_0^z P(x) dx} &= Q(z)e^{\int_0^z P(x) dx} \\ \left(C(z)e^{\int_0^z P(x) dx} \right)' &= Q(z)e^{\int_0^z P(x) dx} \end{aligned}$$

We can now express $C(z)$ as

$$C(z) = e^{-\int_0^z P(x) dx} \int Q(z)e^{\int_0^z P(x) dx} dz.$$

Solution

As suggested, multiplying both sides of the recurrence by z^n and then summing over the index n yields

$$\sum_{n=0}^{\infty} nC_n z^n = \sum_{n=0}^{\infty} n(n+1)z^n + 2 \sum_{n=0}^{\infty} \left(\sum_{k=1}^n C_{k-1} \right) z^n. \quad (1)$$

Now $\sum_{n \geq 0} nC_n z^n = zC'(z)$. Note that

$$\begin{aligned} \sum_{n=0}^{\infty} n(n+1)z^n &= 2 \sum_{n=0}^{\infty} \binom{n+1}{2} z^n \\ &= \frac{2}{z} \sum_{n=0}^{\infty} \binom{n+1}{2} z^{n+1} \\ &= \frac{2}{z} \sum_{n=2}^{\infty} \binom{n}{2} z^n \\ &= \frac{2}{z} \frac{z^2}{(1-z)^3} \\ &= \frac{2z}{(1-z)^3}. \end{aligned}$$

Finally, note that

$$\begin{aligned} \sum_{n=0}^{\infty} \left(\sum_{k=1}^n C_{k-1} \right) z^n &= \sum_{n=0}^{\infty} z^n \sum_{n=0}^{\infty} C_n z^{n+1} \\ &= \frac{zC(z)}{1-z}. \end{aligned}$$

Hence the functional relation that we are seeking is:

$$C'(z) - \frac{2}{1-z}C(z) = \frac{2}{(1-z)^3}.$$

The integrating factor in our case is

$$e^{-2 \int_{x=0}^z \frac{1}{1-z} dx} = e^{2 \ln(1-z)} = (1-z)^2.$$

Hence $C(z)$ is given by

$$C(z) = \frac{1}{(1-z)^2} \int \frac{2}{1-z} dz = \frac{2}{(1-z)^2} \ln \frac{1}{1-z}.$$

Using the fact that $\frac{z}{(1-z)^2} \ln \frac{1}{1-z} = \sum_{n \geq 0} n(H_n - 1)z^n$, we obtain:

$$[z^n] \frac{2}{(1-z)^2} \ln \frac{1}{1-z} = 2(n+1)(H_{n+1} - 1).$$

Homework Assignment 9-1 (10 Points)

Solve the recurrences:

1. $a_n = 5a_{n-1} - 8a_{n-2} + 4a_{n-3}$ for $n > 2$ with $a_0 = 1$, $a_1 = 2$, and $a_2 = 4$.
2. $a_n = 3a_{n-1} - 3a_{n-2} + a_{n-3}$ for $n > 2$ with $a_0 = a_1 = 0$ and $a_2 = 1$.

Hint: Use might need to use:

$$\frac{1}{(1-z)^{m+1}} = \sum_{n=0}^{\infty} \binom{n+m}{m} z^n.$$

Solution

1. As in the exercise, we have $g(z) = 1 - 5z + 8z^2 - 4z^3 = (1-z)(1-2z)^2$. As before, we use the initial conditions to express $f(z)$:

$$\begin{aligned} f(z) &= g(z) \sum_{n=0}^2 a_n z^n \pmod{z^3} \\ &= (1 - 5z + 8z^2 - 4z^3)(1 + 2z + 4z^2) \pmod{z^3} \\ &= 1 - 3z + 2z^2 \\ &= (1-z)(1-2z). \end{aligned}$$

Now $a(z) = 1/(1-2z) = \sum_{n \geq 0} (2z)^n$, so that $a_n = 2^n$ for all $n \geq 0$.

2. Compute $g(z) = 1 - 3z + 3z^2 - z^3 = (1-z)^3$. Using the initial conditions, we can write $f(z)$ as:

$$\begin{aligned} f(z) &= (1-z)^3 z^2 \pmod{z^3} \\ &= z^2. \end{aligned}$$

Thus $a(z) = z^2/(1-z)^3$. We can write this as

$$\frac{z^2}{(1-z)^3} = \frac{\alpha}{1-z} + \frac{\beta}{(1-z)^2} + \frac{\gamma}{(1-z)^3}.$$

Substituting $z = 0, -1, -2$, we obtain the following system of equations:

$$\begin{aligned} \alpha + \beta + \gamma &= 0 \\ 4\alpha + 2\beta + \gamma &= 1 \\ 9\alpha + 3\beta + \gamma &= 4 \end{aligned}$$

from which we obtain: $\alpha = 1$, $\beta = -2$ and $\gamma = 1$. Thus

$$\begin{aligned} a(z) &= \frac{1}{1-z} - \frac{2}{(1-z)^2} + \frac{1}{(1-z)^3} \\ &= \sum_{n=0}^{\infty} z^n - 2 \sum_{n=0}^{\infty} \binom{n+1}{1} z^n + \sum_{n=0}^{\infty} \binom{n+2}{2} z^n. \end{aligned}$$

Hence $a_n = 1 - 2\binom{n+1}{1} + \binom{n+2}{2}$ for $n \geq 3$. This simplifies to $\binom{n}{2}$.

Homework Assignment 9-2 (10 points)

Use generating functions to solve the recurrence:

$$na_n = (n-2)a_{n-1} + 2 \text{ for } n > 2 \text{ with } a_1 = 1, a_2 = 1.$$

Solution

First note that the recurrence actually holds for $n = 2$. Define

$$A(z) = \sum_{n=1}^{\infty} a_n z^n.$$

Multiply both sides of the recurrence by z^n and sum from $n = 2$ onwards to obtain:

$$\sum_{n=2}^{\infty} na_n z^n = \sum_{n=2}^{\infty} (n-2)a_{n-1} z^n + 2 \sum_{n=2}^{\infty} z^n. \quad (2)$$

Note that $zA'(z) = \sum_{n=1}^{\infty} na_n z^n$ so that the left-hand side of equation (2) is $zA'(z) - z$. Write down the right-hand side of recurrence (2) as:

$$\sum_{n=2}^{\infty} na_{n-1} z^n - 2 \sum_{n=2}^{\infty} a_{n-1} z^n + 2 \left(\frac{1}{1-z} - (1+z) \right). \quad (3)$$

Note that $(zA(z))' = \sum_{n=1}^{\infty} (n+1)a_n z^n$ and that multiplying both sides by z yields:

$$z(A(z) + zA'(z)) = \sum_{n=1}^{\infty} (n+1)a_n z^{n+1} = \sum_{n=2}^{\infty} na_{n-1} z^n,$$

which the first term of expression (3). The second term of expression (3) is $zA(z)$. Thus the functional relation we seek is given by:

$$zA'(z) - z = zA(z) + z^2 A'(z) - 2zA(z) + 2 \frac{z^2}{1-z}.$$

This can be simplified to:

$$A'(z) + \frac{1}{1-z} A(z) = \frac{1+z}{(1-z)^2}.$$

The integrating factor in this case is

$$\exp\left(\int_0^z \frac{1}{1-x} dx\right) = \exp(-\ln(1-z)) = \frac{1}{1-z}.$$

Multiplying by the integrating factor and simplifying, we obtain:

$$A(z) = (1-z) \int \frac{1+z}{(1-z)^3} dz.$$

We can solve the integral easily by substituting $x = 1-z$. This transformation gives us $dz = -dx$ and the integral that we have to evaluate is: $\int \frac{x-2}{x^3} dx = \frac{1}{x^2} - \frac{1}{x}$. Thus

$$A(z) = \frac{1}{1-z} - 1 = \sum_{n=0}^{\infty} z^n - 1.$$

Thus for $n \geq 1$, $a_n = 1$.