**RWTH Aachen** **SS 2012**
**Lehrgebiet Theoretische Informatik** Tutorial 12
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# Analysis of Algorithms — Tutorial

Today's tutorial will be based on the following GF:

$$U(z) := \frac{1 - z - \sqrt{(1-3z)(1+z)}}{2z}.$$

## Problem 12-1

Find a closed formula for $f(n)$ such that $[z^n]U(z) \sim f(n)$

**Solution:** The dominant singularity is located at 1/3. It is an algebraic singularity, since

$$U(z) = \frac{1-z}{2z} - (1-3z)^{1/2}\frac{\sqrt{1+z}}{2z} =: h(z) + (1-3z)^{1/2}g(z),$$

where $h(z)$ and $g(z)$ are analytic in 1/3. Unfortunately, $h(z)$ is not analytic on the disc $|z| \le 1/3$ and thus we cannot apply the theorem. To get out of this dilemma, we can use $U'(z) := 2zU(z)$. Here, we know that $U'(z)$ and $U(z)$ only differ by an index-shift and a constant factor of two. Now, $U'(z)$ can be written as:

$$U'(z) = 1 - z - (1-3z)^{1/2}\sqrt{1+z} =: h'(z) + (1-3z)^{1/2}g'(z).$$

$U'(z)$ has the same dominant singularity $\alpha = 1/3$, but $h'(z)$ and $g'(z)$ are analytic on $|z| \le 1/3$. The singularity has order $c = 1/2$. Thus we can apply the theorem, which yields:

$$[z^n]U'(z) \sim \frac{g'(\alpha)n^{-c-1}}{\Gamma(-c)\alpha^n} = \frac{\sqrt{1/3}}{\sqrt{\pi}}3^n n^{-3/2}$$

Then (reverting the index shift and the constant factor 2)

$$[z^n]U(z) \sim \frac{\sqrt{3}}{2\sqrt{\pi}}3^n n^{-3/2}$$

## Problem 12-2

Let $V(z) := -\sqrt{(1-3z)(1+z)}$. What is the relation of $[z^n]V(z)$ and $[z^n]U(z)$?

**Solution:** Since $V(z) = 2zU(z) - 1 + z$ we have

$$[z^n]V(z) = 2[z^{n-1}]U(z) - (n=0) + (n=1).$$

In other words, setting $v_n = [z^n]V(z)$ and $u_n = [z^n]U(z)$:

$$v_n = \begin{cases} 2u_{n-1} & \text{if } n > 1 \\ 2u_0 + 1 & \text{if } n = 1 \\ -1 & \text{if } n = 0 \end{cases}$$

Testing these with the `maxima` tool confirms the formula for small values:

```
(C1) V(z):=-sqrt((1-3*z)*(1+z));

(D1)                   V(z) := - SQRT((1 - 3 z) (1 + z))
(C2) U(z):=(1-z+V(z))/(2*z);

                                    1 - z + V(z)
(D2)                         U(z) := ------------
                                        2 z
(C3) taylor(U(z),z,0,9);

              2      3      4      5       6       7        8        9
(D3)/T/ z + z  + 2 z  + 4 z  + 9 z  + 21 z  + 51 z  + 127 z  + 323 z  + . . .

(C4) taylor(V(z),z,0,8);

                      2      3      4      5       6       7        8
(D4)/T/ - 1 + z + 2 z  + 2 z  + 4 z  + 8 z  + 18 z  + 42 z  + 102 z  + . . .
```

**Problem 12-3**
Compute $W(z) := V(z) + 2/\sqrt{3}\sqrt{1-3z}$ and estimate $[z^n]W(z)$ *as good as possible.*
**Solution:**

$$W(z) = 2/\sqrt{3}\sqrt{1-3z} - \sqrt{1-3z}\sqrt{1+z} = \sqrt{1-3z}(2/\sqrt{3} - \sqrt{1+z}).$$

Using $h(z) = 0$ and $g(z) = 2/\sqrt{3} - \sqrt{1+z}$, we could already apply the theorem. Unfortunately, $g(1/3) = 0$, and thus the theorem only yields the rather bad estimation $[z^n]W(z) = o(3^n n^{-3/2})$. We therefore rewrite $W(z)$ as

$$W(z) = (1-3z)^{3/2}\,\frac{2/\sqrt{3} - \sqrt{1+z}}{1-3z} =: (1-3z)^{3/2} g(z).$$

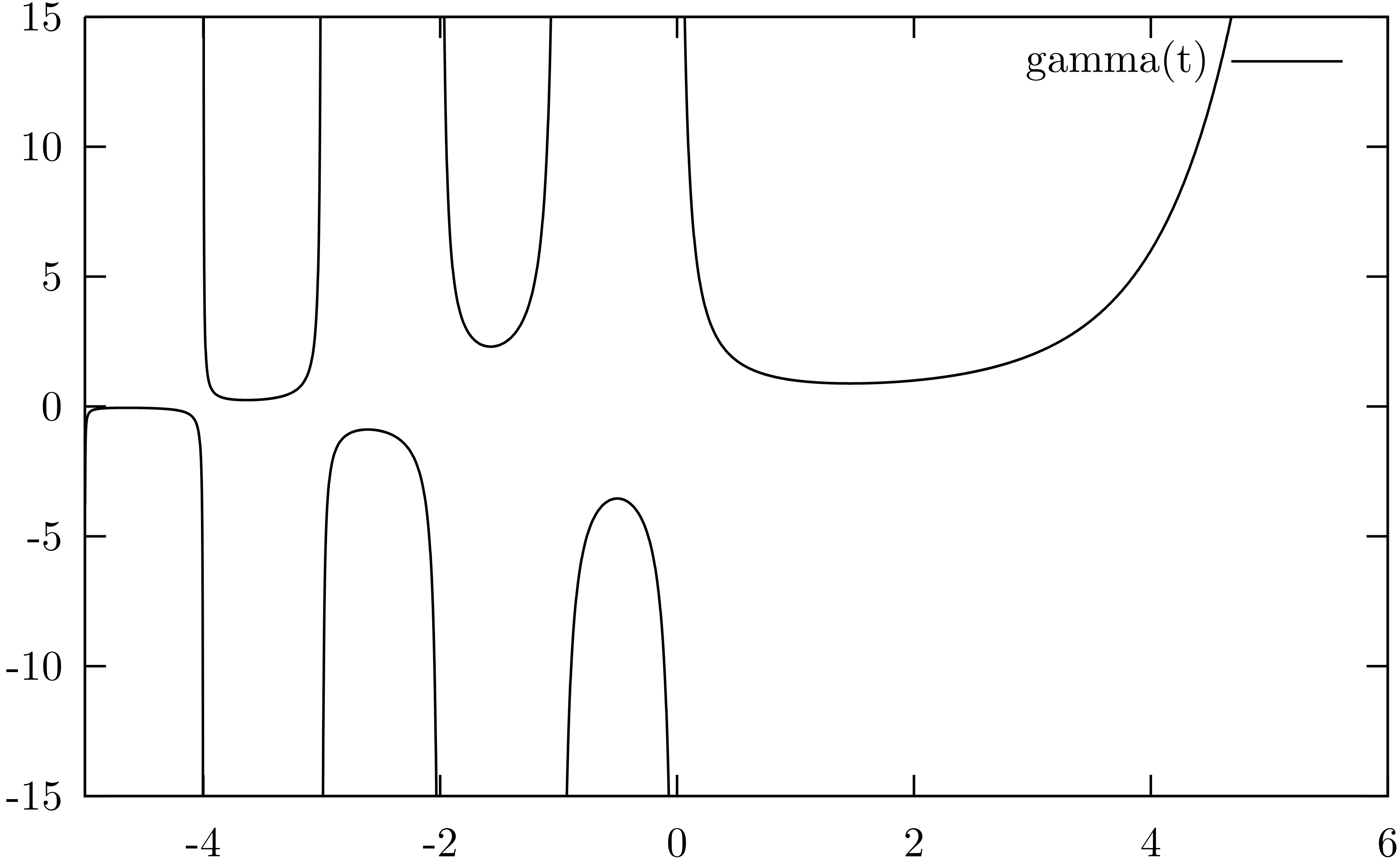
Since $g(z)$ is analytic on $|z| \le 1/3$, $1/3$ is a singularity of order $3/2$ for the whole expression and we can apply the theorem to get a better estimation of $[z^n]W(z)$ (using L'Hospital for $g(1/3)$):

$$[z^n]W(z) = \frac{1}{2\sqrt{1+1/3}}\,\frac{1}{3}\,\frac{n^{-5/2}3^n}{\Gamma(-3/2)} + o(3^n n^{-5/2}) = \frac{\sqrt{3}}{16\sqrt{\pi}}\frac{3^n}{n^{5/2}} + o(3^n n^{-5/2})$$

For your convenience, here are some number that show the quality of above estimation: The exact value for $[z^{50}]W(z)$ is 2568001833006140416, while the above estimation yields approx. $2.480 \times 10^{18}$.

**Problem 12-4**
What follows for $[z^n]V(z)$?
**Lösung:**

$$[z^n]V(z) = -[z^n]\frac{2}{\sqrt{3}}\sqrt{1-3z} + [z^n]W(z) = -\frac{2}{\sqrt{3}}\binom{1/2}{n}3^n + \frac{\sqrt{3}}{16\sqrt{\pi}}\frac{3^n}{n^{5/2}} + o(3^n n^{-5/2})$$



gamma(t)
15
10
5
0
-5
-10
-15
-4
-2
0
2
4
6