RWTH Aachen **SS 2012**
Lehrgebiet Theoretische Informatik Tutorial 9
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Analysis of Algorithms — Tutorial

Problem 9-1

Compute the generating functions of the following series:

1.	$a_n = 2^n + 3^n$	2.	$b_n = (n+1)2^{n+1}$	3.	$c_n = \alpha^n \binom{k}{n}$
4.	$d_n = n - 1$	5.	$e_n = (n+1)^2$		

Solution:

1. The generating function of (α^n) is $\sum_{n\geq 0} \alpha^n z^n$, which yields $\frac{1}{1-\alpha z}$ in closed form. The generating function of $a_n = 2^n + 3^n$ is thus simply $\frac{1}{1-2z} + \frac{1}{1-3z}$.
2. We start with (2^n) and $\frac{1}{1-2z}$. Derivating yields $b_n = (n+1)2^{n+1}$ with generating function $\frac{2}{(1-2z)^2}$.
3. The series $\binom{k}{n}$ has the generating function $(1+z)^k$. Scaling with α results in $c_n = \alpha^n \binom{k}{n}$ with corresponding generating function $(1+\alpha z)^k$.
4. We already know that the series $(n+1) = 1, 2, 3, 4, \ldots$ belongs to the generating function $\frac{1}{(1-z)^2}$. In order to obtain $d_n = -1, 0, 1, 2, 3, \ldots$, we first shift this twice to the right. This yields $0, 0, 1, 2, 3, 4, \ldots$ with generating function $\frac{z^2}{(1-z)^2}$. Now we subtract $1, 0, 0, \ldots$ and obtain d_n with generating function $\frac{z^2}{(1-z)^2} - 1$.
5. Recall that $(n+1) = 1, 2, 3, 4, \ldots$ has the generating function $\frac{1}{(1-z)^2}$. We shift to the right and obtain (n) as well as $\frac{z}{(1-z)^2}$. Derivating yields the desired series $e_n = (n+1)^2$ with generating function $\frac{z+1}{(1-z)^3}$.

Problem 9-2

Compute:

$$\text{(a) } [z^n]\frac{1}{1+2z} \qquad \text{(b) } [z^n]\frac{z+1}{z-1} \qquad \text{(c) } [z^n]\left(\frac{z+1}{z-1}\right)^2 \qquad \text{(d) } [z^n]\frac{1}{\sqrt[3]{5+z}}$$

Solution:

$$\text{(a) } (-2)^n \qquad \text{(b) } -2 + (n=0) \qquad \text{(c) } 4n + (n=0) \qquad \text{(d) } \frac{1}{\sqrt[3]{5}}\binom{-\frac{1}{3}}{n}5^{-n}$$

Homework Assignment 9-1 (10 points)
Find $[z^n]$ for each of the following ordinary generating functions:

$$(1)\ [z^n]\frac{1}{(1-3z)^4} \qquad (2)\ [z^n](1-z)^2\ln\frac{1}{1-z} \qquad (3)\ [z^n]\frac{1}{(1-2z^2)^2}$$

1. One may write $1/(1-3z)^4$ as

$$\begin{aligned}\frac{1}{(1-3z)^4} &= (1-3z)^{-4}\\ &= \sum_{n=0}^{\infty}\binom{-4}{n}(-3z)^n\\ &= \sum_{n=0}^{\infty}\binom{-4}{n}(-3)^n z^n.\end{aligned}$$

The coefficient of z^n is therefore $\binom{-4}{n}(-3)^n$.

2. The Taylor series expansion of $\ln(1-z)$ is

$$\ln(1-z) = -\sum_{n=1}^{\infty}\frac{z^n}{n}.$$

Hence,

$$\begin{aligned}(1-z)^2\ln\frac{1}{1-z} &= (1-z)^2\sum_{n=1}^{\infty}\frac{z^n}{n}\\ &= (1-2z-z^2)\sum_{n=1}^{\infty}\frac{z^n}{n}\\ &= \sum_{n=1}^{\infty}\frac{z^n}{n}-\sum_{n=1}^{\infty}\frac{2z^{n+1}}{n}-\sum_{n=1}^{\infty}\frac{z^{n+2}}{n}\\ &= \sum_{n=1}^{\infty}\frac{z^n}{n}-\sum_{n=2}^{\infty}\frac{2z^n}{n-1}-\sum_{n=3}^{\infty}\frac{z^n}{n-2}\\ &= z-\frac{3}{2}z^2+\sum_{n=3}^{\infty}\left(\frac{1}{n}-\frac{2}{n-1}-\frac{1}{n-2}\right)z^n.\end{aligned}$$

The coefficients of z^0, z^1, and z^2 are, respectively, 0, 1, and $-3/2$. For $n \geq 3$, the coefficient of z^n is

$$\frac{1}{n}-\frac{2}{n-1}-\frac{1}{n-2}.$$

3. As in exercise (1),

$$\begin{aligned}\frac{1}{(1-2z^2)^2} &= (1-2z^2)^{-2}\\ &= \sum_{n=0}^{\infty}\binom{-2}{n}(-2z^2)^n\\ &= \sum_{n=0}^{\infty}\binom{-2}{n}(-2)^n z^{2n}.\end{aligned}$$

Therefore

$$[z^n]\frac{1}{(1-2z^2)^2} = \begin{cases} 0, & \text{if } n \text{ is odd;} \\ \binom{-2}{n/2}(-2)^{n/2}, & \text{if } n \text{ is even.} \end{cases}$$

Homework Assignment 9-2 (10 Points)

Compute the corresponding series of the following generating functions:

1. $A(z) = 3^z$
2. $B(z) = 1/\sqrt{1-z/2}$
3. $C(z) = (1+z)/(1-z)$

Solution:

1. $A(z) = 3^z = e^{z \ln 3} = \sum_{0 \le n} \frac{(\ln 3)^n z^n}{n!}$. The corresponding series is $1, \ln 3, \frac{(\ln 3)^2}{2}, \frac{(\ln 3)^3}{3!} \ldots$.
2. We apply the Newton formula and scale with $-1/2$, which yields $(-\frac{1}{2})^n\binom{-\frac{1}{2}}{n}$. This can be rewritten as:
$$\begin{aligned} (-\frac{1}{2})^n\binom{-\frac{1}{2}}{n} = (-\frac{1}{2})^n(-\frac{1}{2})^{\underline{n}} &= \frac{1}{n!4^n}1 \cdot 3 \cdot 5 \cdots (2n-1) \\ = \frac{(2n)!}{n!4^n}\frac{1}{2^n n!} &= \binom{2n}{n}\frac{1}{8^n} \end{aligned}$$
3. We add the series $1, 1, 1, \ldots$ and the same series shifted to the right $0, 1, 1, 1, \ldots$. This yields $1, 2, 2, 2, \ldots$.