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SS 2012
Tutorial 8
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Analysis of Algorithms — Tutorial

Problem 8-1

```
f(int n){
  int s=0;
  if (n==0) return 1;
  for (int i=0;i<n;i++)
    s+=f(i);
  return s;
}
```

Compute how often the 5th line of this program is executed using generating functions.

Solution

Let A_n denote the how often the 5th line is called on input n. We immediately obtain $A_0 = 0$ and $A_n = n + \sum_{i=0}^{n-1} A_i$.
The corresponding generating function is thus

$$A(z) = \frac{z}{(1-z)^2} + \frac{zA(z)}{1-z}.$$

This implies

$$A(z)(1-2z) = \frac{z}{1-z}$$

and therefore

$$\begin{aligned} A(z) &= \frac{1}{(1-z)(1-2z)} z \\ &= \left(\frac{A}{1-z} + \frac{B}{1-2z} \right) z \end{aligned}$$

for some A, B. Setting $z = 0$ implies $A + B = 1$ and $z = -1$ implies $A/2 + B/3 = 1/6$. We easily obtain $A = -1$ and $B = 2$ and therefore

$$A(z) = \left(\frac{1}{1-z} + \frac{2}{1-2z} \right) z.$$

Since the multiplication with z is just a shift, the solution is therefore

$$a_{n+1} = -1 + 2 \cdot 2^n = 2^{n+1} - 1$$

Problem 8-2
Prove that

$$(1+z)^r = \sum_{n=0}^{\infty} \binom{r}{n} z^n,$$

for $n \in \mathbf{N}$ and $r \in \mathbf{R}_{\geq 0}$.

Solution
Let $f(z) = (1+z)^r$. The k-th derivative of f is

$$f^{(k)} = r^{\underline{k}}(1+z)^{r-k},$$

since $r \geq 0$. Using the Taylor expansion, we obtain

$$f(z) = \sum_{n=0}^{\infty} \frac{f^{(k)}(0)}{n!} z^n = \sum_{n=0}^{\infty} \frac{r^{\underline{n}}}{n!} z^n.$$

Since

$$\frac{r^{\underline{n}}}{n!} = \binom{r}{n},$$

we obtain the claimed result.

Homework Assignment 8-1 (10 Points)
Solve this recurrence using generating functions:

$$a_n = 2a_{n-1} + 3a_{n-2}$$

and $a_0 = 0$, $a_1 = 2$.

Solution
We start with $S(z) = \sum_{n\geq 0} a_n z^n$. Since the values a_0 and a_1 are given, we rewrite this as

$$S(z) = a_0 + a_1 z + \sum_{n\geq 2} a_n z^n.$$

Using the recursive definition, this yields

$$S(z) = a_0 + a_1 z + \sum_{n\geq 2}(2a_{n-1} + 3a_{n-2})z^n = a_0 + a_1 z + \sum_{n\geq 2} 2a_{n-1}z^n + \sum_{n\geq 2} 3a_{n-2}z^n.$$

Shifting results in

$$\begin{aligned} S(z) &= a_0 + a_1 z + z\sum_{n\geq 1} 2a_n z^{n+1} + \sum_{n\geq 0} 3a_n z^{n+2} \\ &= a_0 + a_1 z - 2a_0 z + 2z\sum_{n\geq 0} a_n z^n + 3z^2 \sum_{n\geq 0} a_n z^n \\ &= 2z + 2zS(z) + 3z^2 S(z) \end{aligned}$$

This implies

$$S(z) \quad = \quad \frac{2z}{1-2z-3z^2} = \frac{2z}{(1+z)(1-3z)}$$

Now, we need to find a and b such that

$$S(z) = \frac{2z}{(1+z)(1-3z)} = \frac{a}{1+z} + \frac{b}{1-3z}.$$

Setting $z = 0$ and $z = 1$ implies $a + b = 0$ as well as $-2a + 2b = 2$, and thus $a = -1/2$ and $b = 1/2$. We obtain

$$S(z) = -\frac{1/2}{1+z} + \frac{1/2}{1-3z} = -\frac{1}{2}\sum_{n\geq 0}(-1)^n z^n + \frac{1}{2}\sum_{n\geq 0} 3^n z^n$$

Thus, we have

$$a_n = \frac{1}{2}3^n - \frac{1}{2}(-1)^n.$$