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SS 2012
Problem Set 4
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Analysis of Algorithms — Tutorial

Problem 4-1

The following program prints a lot of numbers. How many?

```
void snapl(int n) {
  if(n ≡ 0 || n ≡ 2) return;
  if(n ≡ 1) printf("8156\n");
  else {
    snapl(n − 2);
    snapl(n − 3);
    snapl(n − 2);
    snapl(n − 3);
    snapl(n − 2);
  }
}
```

Solution:

We get the recurrence $a_n = 3a_{n-2} + 2a_{n-3}$ for $n > 3$ and $a_0 = a_2 = 0$, $a_1 = 1$.
The characteristic polynomial is $z^3 - 3z - 2$. We need to find its roots.
Obviously, -1 is a root of $z^3 - 3z - 2$. By polynomial division we obtain $z^2 - z - 2$ as the remaining polynomial. Since -1 is also a root of this polynomial, we need to apply polynomial division again, which yields $z - 2$. The remaining root is 2.
The closed representation for a_n is of the form $\lambda(-1)^n + n\mu(-1)^n + \nu 2^n$. Using a_0, a_1 and a_2, we obtain

$$\begin{aligned} a_0 = 0 &= \lambda + \nu && (1)\\ a_1 = 1 &= -\lambda - \mu + 2\nu && (2)\\ a_2 = 0 &= \lambda + 2\mu + 4\nu. && (3) \end{aligned}$$

The first equation implies $\lambda = -\nu$ and thus

$$\begin{aligned} a_1 = 1 &= -3\lambda - \mu && (4)\\ a_2 = 0 &= -3\lambda + 2\mu. && (5) \end{aligned}$$

A simple computation now shows $\mu = -\frac{1}{3}$, which implies $3\lambda = 2\mu$ and thus $\lambda = -\frac{2}{9}$. The closed form is therefore

$$-\frac{2}{9}(-1)^n - n\frac{1}{3}(-1)^n + \frac{2}{9}2^n = \frac{2+3n}{9}(-1)^{n+1} + \frac{2^{n+1}}{9}.$$

Problem 4-2

Solve the following recurrence: Let $b_1 = b_2 = b_3 = 1$ and

$$b_n = 3b_{n-1} - 4b_{n-2} + 12b_{n-3} \text{ for } n > 3.$$

Solution:

Here the characteristic polynomial is $x^3 - 3x^2 + 4x - 12$. We guess the first root $x_0 = 3$. Using polynomial division, we gain $x^2 + 4$. The two other roots are thus $x_1 = 2i$ and $x_2 = -2i$. Hence, the solution is of form $b_n = \lambda 3^n + \mu_1(2i)^n + \mu_2(-2i)^n$. Using the known values for $n = 1, 2, 3$, we obtain a system of equations with three variables λ, μ_1, μ_2 that can be solved with the standard methods.

Homework Assignment 4-1 (10 Points)

Solve the following recurrence: Let $a_0 = 0$, $a_1 = 3$ and

$$a_n = 4a_{n-1} - 4a_{n-2} \text{ for } n > 1.$$

Solution: The characteristic polynomial is $x^2 - 4x + 4 = (x - 2)^2$ with the only root $x_0 = 2$. The solution is therefore of form $a_n = \lambda 2^n + \mu n 2^n$. Our constraints yield $a_0 = 0 = \lambda$ and $\mu = \frac{a_1}{2} = \frac{3}{2}$.

Homework Assignment 4-2 (10 Points)

Solve the following recurrence and find a nice representation of the solution (in a mathematical sense).

$$\begin{aligned} c_0 &= 2 \\ c_1 &= 4 \\ c_n &= c_{n-2}^{\log c_{n-1}} \end{aligned}$$

Solution: We apply the logarithm and obtain

$$\log c_n = \log c_{n-1} \cdot \log c_{n-2}.$$

In order to obtain a sum instead of a product, we repeat this and obtain

$$\log\log c_n = \log\log c_{n-1} + \log\log c_{n-2}.$$

Substituting $d_n = \log\log c_n$ yields

$$\begin{aligned} d_0 &= 0 \\ d_1 &= 1 \\ d_n &= d_{n-1} + d_{n-2} \end{aligned}$$

Since this describes the Fibonacci numbers, we obtain $c_n = 2^{2^{F_n}}$, where F_n denotes the n-th Fibonacci number.