Lower Bounds on the Complexity of MSO_1 Model-Checking

Somnath Sikdar

Joint work with

Robert Ganian Petr Hliněný Alexander Langer

Jan Obdržálek Peter Rossmanith

Theoretical Computer Science,
RWTH Aachen University, Germany.

Faculty of Informatics,
Masaryk University, Brno, Czech Republic.

Outline

Algorithmic Meta Theorems

Theorems that identify tractable **problem classes**.

Example

- All graph properties expressible in MSO_2 can be decided in linear time on graphs of bounded treewidth [Courcelle, 1990].
- All problems in MAX SNP have constant-factor approximation algorithms [Papadimitriou and Yannakakis, 1991].
- Compact parameterized problems expressible in CMSO admit polynomial kernels on graphs of bounded genus [Bodlaender et al, 2010].

Uses

- Quick way of checking whether a problem admits an algorithm of a particular kind.

Courcelle's Theorem

(rephrased in the parlance of parameterized complexity)

Theorem (Courcelle, 1990)

Let $\varphi \in MSO_2$ and let $\mathcal{C}$ be the class of all graphs. Then MSO_2 model-checking problem $MC(MSO_2, \mathcal{C})$: "Does $G \models \varphi$?" is fixed-parameter tractable wrt the parameter $|\varphi| + tw(G)$.

No **lower bounds** were known till recently.

Courcelle's Theorem: Lower Bounds

Are there classes of **unbounded treewidth** for which Courcelle's Theorem holds?

YES!

Graph classes with very slowly growing treewidth ($\log^* n$, for instance).

Question

How fast must the treewidth grow for Courcelle's Theorem to fail?

Kreutzer and Tazari show that Courcelle's Theorem fails for graph classes with moderately unbounded treewidth.

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Graph Classes with Moderately Unbounded Treewidth

Definition (Kreutzer and Tazari)

The treewidth of a graph class $\mathcal{C}$ is polylogarithmically unbounded if for all $c > 1$ the following holds: for all $n \in \mathbf{N}$ there exists $G_n \in \mathcal{C}$ with

- $\log^c(|G_n|) \leq tw(G_n)$ *(unboundedness);*
- $|G_n| = 2^{o(n)}$ *(small size);*
- *G_n can be constructed in time $2^{o(n)}$ (constructibility).*

Courcelle's Theorem: A Lower Bound

Theorem (Kreutzer and Tazari, 2010)

Let $\mathcal{C}$ be a graph class that is

- *closed under subgraphs, and*
- *has polylogarithmically unbounded treewidth.*

Then given

- *$G \in \mathcal{C}$, $\varphi \in MSO$ with $|\varphi|$ as parameter,*

deciding whether $G \models \varphi$ is not in XP, unless SAT can be solved in subexponential time.

High-level Proof Idea

Reduce Sat to MC(MSO$_2$, $\mathcal{C}$).

- *Input:* A SAT formula F of length n.
- *Question:* Is F satisfiable?

Reduction

1. Construct $G_n \in \mathcal{C}$ s.t. $\log^c(|G_n|) < \text{tw}\,(G_n)$ and $|G_n| = 2^{o(n)}$.
2. Encode F in a subgraph of G_n.
   - Using closure under subgraphs.
3. Define an MSO-formula φ (independent of F) s.t. F satisfiable iff $G_n \models \varphi$.
   - Deciding $G_n \models \varphi$ in XP takes time $2^{o(n)\cdot f(|\varphi|)}$, subexponential in $|F|$.

Aspects of Kreutzer & Tazari's Theorem

- Threshold for treewidth is more-or-less strict.
  - $\exists$ subgraph-closed classes with $\text{tw}(G) = \log |G|$ that can be model-checked in XP-time [Makowski and Mariño, 2003].
- The proof requires certain witnesses to be constructed efficiently.
  - Constructibility is part of the definition.
  - Proofs are very technical and spread over several papers.

Outline

Main Theorem

Theorem

Let $\mathcal{C}$ be a graph class that is

- *closed under subgraphs;*
- *has polylogarithmically unbounded treewidth.*

Then the MSO_1 model-checking problem on vertex labeled graphs from $\mathcal{C}$ is not in XP, unless 3-Colorability is in time $2^{o(n)}$ with subexponential advice.

- The labels are from a fixed, finite set.
- **Nonuniform ETH**: SAT, 3-Colorability are not in $2^{o(n)}$ time with subexponential advice.

Major Differences Between the Two Results

1. We use a different logic.
    - **Our result:** applies to MSO_1 model-checking on vertex-labeled graphs.
    - **K & T's result:** applies to MSO_2 model-checking on unlabeled graphs.

   The two logic classes not comparable: consider Hamiltonian Cycle and Red Blue Dominating Set.
2. We assume that witnesses are given as advice:
    - No constructibility requirement;
    - Stronger complexity assumption: Nonuniform ETH;
    - Since our proof does not require constructibility, it is much shorter and easier.

On the Constructibility Clause

Our definition of polylogarithmically unbounded treewidth:

Definition

The treewidth of a graph class $\mathcal{C}$ is polylogarithmically unbounded if for all $c > 1$ the following holds. For all $n \in \mathbf{N}$ there exists $G_n \in \mathcal{C}$ with

- $\log^c(|G_n|) \leq tw(G_n)$ *(unboundedness);*
- $|G_n| = 2^{o(n)}$ *(small size).*

- No constructibility requirement.

ETH versus Nonuniform ETH (NETH)

Exponential Time Hypothesis [Impagliazzo, Paturi, and Zane, 2001]:

- n-variable 3-SAT cannot be solved in $2^{o(n)}$ time.
- Can be formulated using other problems such as Vertex Cover or 3-Colorability.

NETH: n-variable 3-SAT not solvable in $2^{o(n)}$ time using:

- a family of algorithms, one for each input length;
- a circuit-family $\mathcal{F}$ s.t. for each input length n, $\exists C_n \in \mathcal{F}$ with $|C_n| \leq 2^{o(n)}$;
- an algorithm that receives oracle advice which depends only on the input length n and has $2^{o(n)}$ bits.

Can be formulated in terms of Vertex Cover or 3-Colorability.

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Main Theorem

Theorem

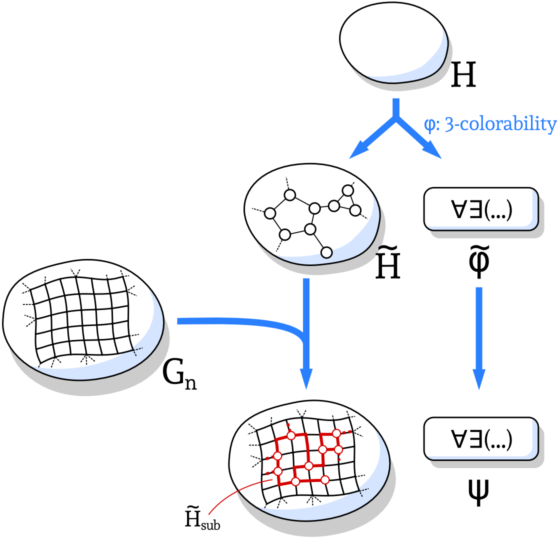
Let $\mathcal{C}$ be a graph class s.t.

- *$\mathcal{C}$ is closed under subgraphs;*
- *$\mathcal{C}$ has polylogarithmically unbounded treewidth.*

Then the MSO_1 model-checking problem on vertex labeled graphs from $\mathcal{C}$ is not in XP, unless 3-Colorability is in time $2^{o(n)}$ with subexponential advice.

Proof. A multistage reduction from 3-Colorability.

Proof Outline



Step 1: Reducing to a Subcubic Planar Graph

Given

- φ: MSO_1 formula expressing 3-Colorability.
- H: n-vertex graph, instance of 3-Colorability.

Reduce $(H, \varphi) \rightarrow (\widetilde{H}, \tilde{\varphi})$ in polynomial-time "preserving" parameters:

- **Equivalence**: $H \models \varphi$ iff $\widetilde{H}_{sub} \models \tilde{\varphi}$ for every subdivision $\widetilde{H}_{sub}$ of $\widetilde{H}$.
- **Parameter-Preserving**: $\tilde{\varphi}$ depends only on φ and $|\tilde{\varphi}| = O(|\varphi|)$;
- H is $\{1, 3\}$-planar.

$\widetilde{H}$ may not be in the class $\mathcal{C}$ but we want a graph in $\mathcal{C}$ that "contains" $\widetilde{H}$.

Step 2: Finding a Graph in $\mathcal{C}$ containing $\widetilde{H}$

$|H| = n$ and $|\widetilde{H}| \leq n^b$, for some constant b.

Polylogarithmic unboundedness of tw $(\mathcal{C})$

- $\exists G \in \mathcal{C}$ s.t. $\log^c |G| \leq \text{tw}(G)$ and $|G| = 2^{n^\epsilon}$.

Grid-like subgraphs [Reed and Wood, 2008]

- $\log^c |G| \leq \text{tw}(G)$ and $|G| = 2^{n^\epsilon}$ implies $n^{O(1)} \leq \text{tw}(G)$.
- $n^{O(1)} \leq \text{tw}(G)$ implies G contains a **grid-like subgraph** Γ_n **of order** n: Γ_n "contains" a subdivision $\widetilde{H}_{\text{sub}}$ of $\widetilde{H}$.

Closure of $\mathcal{C}$ under subgraphs

- $\Gamma_n \in \mathcal{C}$.

Summary so far

- Can "embed" $\widetilde{H}$ in a graph from $\mathcal{C}$ of size $2^{o(n)}$.

Step 3: Using Subexponential Advice

Supexponential advice

- Γ_n has size $2^{o(n)}$ and depends only on n: supplied as advice.

Using vertex labels to identify H_{sub} in Γ_n

- Γ_n "contains" H_{sub}: can construct a vertex labeling λ and a formula $\psi \in \text{MSO}_1[L]$ s.t.

$$H_{\text{sub}} \models \varphi \text{ iff } (\Gamma_n, \lambda) \models \psi.$$

Model-checking $\mathcal{C}$ in XP implies

- deciding $(\Gamma_n, \lambda) \models \psi$ in $|\Gamma_n|^{f(|\psi|)}$ time;
- thereby deciding $H \models \varphi$ in $|\Gamma_n|^{f(|\psi|)} = 2^{o(n) \cdot f(|\psi|)} = 2^{o(n)}$ time, contradicting nonuniform ETH.

Outline

Summary

Main Contribution

- Strengthen and simplify Kreutzer and Tazari's impressive result.

Extending to Unlabeled MSO_1?

- **Open.** Is there is a (nontrivial) graph class where model-checking MSO_1 is easy but $\text{MSO}_1[L]$ is hard?
- This indicates that the result might be extendable to unlabeled MSO_1.

Thank You!