Lower Bounds on the Complexity of MSO₁ Model-Checking

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### Outline

1. **Motivation**
2. **Main Theorem**
3. **Proof Overview**
4. **Consequences**
Algorithmic Meta Theorems

Theorems that identify classes of tractable problems, rather than a few isolated problems.

Examples

- All graph properties expressible in MSO$_2$ can be decided in linear time on graphs of bounded treewidth [Courcelle, 1990].
- All problems in MAX SNP have constant-factor approximation algorithms [Papadimitriou and Yannakakis, 1991].
- Compact parameterized problems expressible in CMSO admit polynomial kernels on graphs of bounded genus [Bodlaender et al, 2010].

Uses

- Quick way of checking whether a problem admits an algorithm of a particular kind.
Courcelle’s Theorem

Theorem (Courcelle, 1990)

Any graph property definable in monadic second-order logic with quantification over sets of vertices and/or edges can be decided in linear time on any class of graphs of bounded treewidth.

- Linear-time algorithms for several NP-hard problems on graphs of “small” treewidth: Hamiltonian Cycle, Vertex Cover, 3-Colorability.

**Hamiltonian Cycle** There exists a set $C \subseteq E$ of edges that
- $C$ induces a connected graph in which every vertex has degree exactly two;
- every vertex is in $V(C)$. 
The Model-Checking Problem

Definition ($\mathcal{L}$-Model-Checking)

Let $\mathcal{C}$ be a class of graphs and let $\mathcal{L}$ be a logic. The $\mathcal{L}$-model-checking problem denoted by $MC(\mathcal{L}, \mathcal{C})$ is: given $G \in \mathcal{C}$ and $\varphi \in \mathcal{L}$, decide whether $G \models \varphi$.

If $\mathcal{L} = \text{MSO}_2$ then this is the MSO-model-checking problem.
Courcelle’s Theorem . . .

. . . rephrased in the parlance of parameterized complexity:

**Theorem (Courcelle, 1990)**

Let $\varphi \in MSO_2$ and let $\mathcal{C}$ be the class of all graphs. Then $MSO_2$ model-checking problem $MC(MSO_2, \mathcal{C})$: “Does $G \models \varphi$?” is fixed-parameter tractable wrt the parameter $|\varphi| + \text{tw}(G)$.

Extended to **(directed) graphs with vertex/edge labels** (from a finite set) and **problems involving evaluations of sets** definable in MSO [Arnborg, Lagergren and Seese, 1991].

No **lower bounds** were known till recently.
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Courcelle’s Theorem: Lower Bounds

Are there classes of unbounded treewidth for which Courcelle’s Theorem holds?

**YES!**

Let $C = \{ G \mid \text{tw} (G) = \log^* |G| \}$. Given an MSO-formula $\varphi$ and an $n$-vertex graph $G \in C$, time taken to decide $G \models \varphi$:

$$\exp(|\varphi|) (\text{tw} (G)) \cdot n \leq \exp(|\varphi|) (\text{tw} (G)) \cdot \exp (\log^* n) (\log^* n) \leq n^2,$$

where $\exp^{(0)}(x) = x$ and

$$\exp^{(i)}(x) = 2^{\exp^{(i-1)}(x)}.$$

**Question**

How fast must the treewidth grow for Courcelle’s Theorem to fail?
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**Question**

*How fast must the treewidth grow for Courcelle’s Theorem to fail?*
Courcelle’s Theorem: Lower Bounds ...

**Theorem (Makowsky and Mariño, 2004)**

If $C$ is a class of graphs of unbounded treewidth that is closed under topological minors and $G \in C$, then deciding whether $G \models \varphi$ is not in FPT wrt $|\varphi|$ as parameter unless $P = NP$.

- Closure under topological minors is a very strong restriction.
- Kreutzer and Tazari: Similar result without this restriction for graph classes with **moderately unbounded treewidth**.
Classes of Unbounded Treewidth

Definition (Bounded Treewidth)

Let $f : \mathbb{N} \rightarrow \mathbb{N}$. A class $\mathcal{C}$ of graphs have $f$-bounded-treewidth if for all $G \in \mathcal{C}$, we have that $\text{tw}(G) \leq f(|G|)$.

Examples

- Courcelle’s Theorem: $f(n) := c$, a constant.
- $f(n) := n$ is the maximum function that makes sense.
- In Kreutzer and Tazari: $f(n) := \log^c n$, for some constant $c > 0$. 
Polylogarithmically Unbounded Classes

Definition (Kreutzer and Tazari)

The treewidth of a graph class \( \mathcal{C} \) is **polylogarithmically unbounded** if for all \( c > 1 \) the following holds: for all \( n \in \mathbb{N} \) there exists \( G_n \in \mathcal{C} \) with

- \( \log^c(|G_n|) \leq tw(G_n) \) (unboundedness);
- \( n \leq tw(G_n) \leq n^\gamma \), for some fixed \( \gamma \) (density);
- \( G_n \) can be constructed in time \( 2^{n^\epsilon} \), for some fixed \( \epsilon < 1 \) (constructibility).

Note

\[
\log^c(|G_n|) \leq tw(G_n) \leq n^\gamma \implies |G_n| \leq 2^{n^\gamma/c}.
\]
Courcelle’s Theorem: A Lower Bound

Theorem (Kreutzer and Tazari, 2010)

Let $\mathcal{C}$ be a graph class with the following properties:

- $\mathcal{C}$ is closed under subgraphs;
- the treewidth of $\mathcal{C}$ is polylogarithmically unbounded.

Then $\text{MC}(\text{MSO}_2, \mathcal{C})$ is not in $\text{XP}$ ($|G|^f(|\varphi|)$ for any computable $f$), unless SAT can be solved in subexponential time.
High-level Proof Idea

Reduce $\text{Sat}$ to $\text{MC} (\text{MSO}_2, \mathcal{C})$.

- **Input:** A SAT formula $F$ of length $n$.
- **Question:** Is $F$ satisfiable?

**Reduction**

1. Construct $G_n \in \mathcal{C}$ of treewidth $n^d$ s.t. $\log^c (|G_n|) < \text{tw} (G_n)$ and $c > d$.
   - Conditions 1 and 2: $G_n$ exists in $\mathcal{C}$.
   - Condition 3: $G_n$ is efficiently constructible and $|G_n| < 2^{n^d/c}$.

2. Encode $F$ in a subgraph of $G_n$ (exists because $\text{tw} (G_n) \approx n^d$).
   - Using closure under subgraphs.

3. Define an MSO-formula $\varphi$ (independent of $F$) s.t. $F$ satisfiable iff $G_n \models \varphi$.
   - Deciding $G_n \models \varphi$ in XP takes time $2^{n^{c/d} \cdot f(|\varphi|)}$, subexponential in $|F|$.
A Critique of Kreutzer & Tazari’s Result

- There are classes $C$ closed under subgraphs with logarithmic treewidth s.t. $MC(MSO_2, C)$ is in XP [Makowski and Mariño, 2004].
  - Threshold for treewidth is more-or-less strict.
- The constructibility clause in the definition of polylogarithmically unbounded treewidth is unnatural.
- Proofs are very technical and spread over several papers.
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Main Theorem I

**Theorem**

Let \( C \) be a graph class s.t.
- \( C \) is closed under subgraphs;
- the treewidth of \( C \) is polylogarithmically unbounded.

Then the \( \text{MSO}_1 \) model-checking problem on vertex labeled graphs from \( C \) is not in \( \text{XP} \), unless \( 3\text{-Colorability is in time } 2^{o(n)} \text{ with subexponential advice.} \)

- The labels are from a **fixed**, finite set.
- **Nonuniform ETH**: SAT, 3-Colorability are **not in** \( 2^{o(n)} \) time with subexponential advice.
Major Differences Between the Two Results

1. We use a **weaker logic**.
   - **Our result**: applies to MSO$_1$ model-checking on vertex-labeled graphs.
   - **K & T’s result**: applies to MSO$_2$ model-checking on unlabeled graphs.

2. **No constructibility requirement**.
   - We use a **stronger complexity assumption**: Nonuniform ETH.

3. **Easy proofs!**
MSO\textsubscript{2} versus MSO\textsubscript{1} with Vertex Labels

MSO\textsubscript{1} with vertex labels is weaker than MSO\textsubscript{2}.

- Hamiltonian Path/Cycle cannot be expressed in MSO\textsubscript{1} with vertex labels.

Results such as Courcelle’s Theorem and Courcelle, Makowski and Rotics’s Theorem for rankwidth can be extended to vertex-labeled graphs.

- Extending C,M,R’s Theorem for rankwidth from MSO\textsubscript{1} to MSO\textsubscript{2} would imply $\text{EXP} = \text{NEXP}$.
On the Constructibility Clause

Our definition of polylogarithmically unbounded treewidth:

**Definition**

The treewidth of a graph class $\mathcal{C}$ is **polylogarithmically unbounded** if there is a constant $\gamma$ s.t. for all $c > 1$ the following holds. For all $n \in \mathbb{N}$ there exists $G_n \in \mathcal{C}$ with

- $\log^c(|G_n|) \leq \text{tw}(G_n)$ (unboundedness);
- $n \leq \text{tw}(G_n) \leq n^{\gamma}$ (density).

**Note:** $|G_n| \leq 2^{n^{\gamma/c}}$.

- No constructibility requirement.
- At the expense of a stronger complexity-theoretic assumption: Nonuniform ETH.
ETH versus Nonuniform ETH (NETH)

**Exponential Time Hypothesis** [Impagliazzo, Paturi, and Zane, 2001]:
- $n$-variable 3-SAT cannot be solved in $2^{o(n)}$ time.
- Can be formulated using other problems such as Vertex Cover or 3-Colorability.

**NETH**: $n$-variable 3-SAT not solvable in $2^{o(n)}$ time using:
- a family of algorithms, one for each input length;
- a circuit-family $\mathcal{F}$ s.t. for each input length $n$, $\exists C_n \in \mathcal{F}$ with $|C_n| \leq 2^{o(n)}$;
- an algorithm that receives oracle advice which depends only on the input length $n$ and has $2^{o(n)}$ bits.

Can be formulated in terms of Vertex Cover or 3-Colorability.
Main Theorem II

Our result can be strengthened by assuming that the label set is arbitrary but finite.

**Theorem**

Let $L$ be a finite label set and let $\varphi \in \text{MSO}_1[L]$. Let $C$ be a graph class s.t.

- $C$ is closed under subgraphs;
- the treewidth of $C$ is polylogarithmically unbounded.

Then the $\text{MSO}_1$ model-checking problem on vertex labeled graphs from $C$ is not in XP, unless all problems in PH can be solved in time $2^{o(n)}$ with subexponential advice.
Main Theorem I

Theorem

Let $\mathcal{C}$ be a graph class s.t.

- $\mathcal{C}$ is closed under subgraphs;
- the treewidth of $\mathcal{C}$ is polylogarithmically unbounded.

Then the MSO$_1$ model-checking problem on vertex labeled graphs from $\mathcal{C}$ is not in XP, unless 3-Colorability is in time $2^{o(n)}$ with subexponential advice.

Proof. A multistage reduction from 3-Colorability.
Proof Idea: Stage I

Let \( \varphi' \in \text{MSO}_1 \) express \textbf{3-Colorability} and let \( H' \) be an instance of this problem.

\textbf{Reduce} \( (H', \varphi') \rightarrow (H, \varphi) \) in polynomial time s.t.

- \( H \) is \( \{1, 3\} \)-planar;
- \( \varphi \) depends only on \( \varphi' \) and \( |\varphi| = O(|\varphi'|) \).
- \( H' \models \varphi' \) iff \( H_{sub} \models \varphi \) for every subdivision \( H_{sub} \) of \( H \).

\textbf{Note that}

- \( \varphi \) is an “interpretation” of \textbf{3-Colorability} closed under edge subdivisions;
- \( |H'| = n \) and \( |H| \leq n^b \) for some constant \( b \).
Proof Idea: Grid-Like Subgraphs

**Polylogarithmic Unboundedness of** $\text{tw}(C)$

- $\exists G_n \in C$ s.t. $\text{tw}(G_n) \geq \log^c(|G_n|)$ and $n^{5b} \leq \text{tw}(G_n) \leq n^{5b\gamma}$.
- $|G_n| \leq 2^{n^{5b\gamma}/c}$ for $c > 5b\gamma$.

**Grid-Like Subgraphs** [Reed and Wood, 2008]

- $\text{tw}(G_n) \geq n^{5b}$ implies $G_n$ contains a grid-like subgraph $\Gamma_{n^b}$ of order $n^b$.
- $\Gamma_{n^b}$ “contains” a subdivision $H_{\text{sub}}$ of $H$.

**Closure of** $C$ **under Subgraphs**

- $\Gamma_{n^b} \in C$. 
Proof Idea: Stage II

Lemma

Let $\Gamma_n$ “contain” $K$ and let $\varphi \in \text{MSO}_1$. There is a fixed finite set $L$ s.t. one can in poly time construct a labeling $\lambda : V(\Gamma_n) \to L$ and $\psi \in \text{MSO}_1[L]$ (depends only on $\varphi$) s.t.

$$K \models \varphi \iff (\Gamma_n, \lambda) \models \psi.$$

- Since $\Gamma_n$ “contains” $H_{sub}$, we have:

$$H' \models \varphi' \iff H \models \varphi \iff H_{sub} \models \varphi \iff (\Gamma_n, \lambda) \models \psi.$$

- $|\Gamma_n| \leq 2^{n^{5b/c}}$; supplied as advice of subexponential size.

Time taken to decide $H' \models \varphi'$ is $|\Gamma_n|^{f(|\psi|)} = 2^{o(n)}$. 
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Consequences for Directed Width Measures

Extension of [Ganian et al., 2010].

**Theorem**

Unless NETH fails, there exists no *directed width measure* $\delta$ satisfying following three properties:

1. $\delta$ is *closed under subdigraphs*;
2. $\exists$ *digraph class* $\mathcal{C}$ of *bounded $\delta$-width* with $\text{tw}(\mathcal{C})$ *polylogarithmically unbounded*;
3. for $L$-vertex-labeled digraphs $D$ and $\varphi \in \text{MSO}_1[L]$, *deciding $D \models \varphi$ is in time* $O(|D|^{f(\delta(D),|\varphi|)})$. 
Summary

Main Contribution
- Strengthen and simplify Kreutzer and Tazari’s impressive result.

Extending to Unlabeled MSO$_1$?
- **Open.** Is there a (nontrivial) graph class where model-checking MSO$_1$ is easy but MSO$_1[L]$ is hard?
- This indicates that the result might be extendable to unlabeled MSO$_1$. 
Thank You!