Lower Bounds on the Complexity of MSO_1 Model-Checking

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Joint work with

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Outline

Algorithmic Meta Theorems

Theorems that identify **classes** of tractable problems, rather than a few isolated problems.

Examples

- All graph properties expressible in MSO_2 can be decided in linear time on graphs of bounded treewidth [Courcelle, 1990].
- All problems in MAX SNP have constant-factor approximation algorithms [Papadimitriou and Yannakakis, 1991].
- Compact parameterized problems expressible in CMSO admit polynomial kernels on graphs of bounded genus [Bodlaender et al, 2010].

Uses

- Quick way of checking whether a problem admits an algorithm of a particular kind.

Courcelle's Theorem

Theorem (Courcelle, 1990)

Any graph property definable in monadic second-order logic with quantification over sets of vertices and/or edges can be decided in linear time on any class of graphs of bounded treewidth.

- Linear-time algorithms for several NP-hard problems on graphs of "small" treewidth: Hamiltonian Cycle, Vertex Cover, 3-Colorability.

Hamiltonian Cycle There exists a set $C \subseteq E$ of edges that

- C induces a connected graph in which every vertex has degree exactly two;
- every vertex is in $V(C)$.

The Model-Checking Problem

Definition ($\mathcal{L}$-Model-Checking)

Let $\mathcal{C}$ be a class of graphs and let $\mathcal{L}$ be a logic. The $\mathcal{L}$-model-checking problem denoted by $MC(\mathcal{L}, \mathcal{C})$ is: given $G \in \mathcal{C}$ and $\varphi \in \mathcal{L}$, decide whether $G \models \varphi$.

If $\mathcal{L} = \mathrm{MSO}_2$ then this is the MSO-model-checking problem.

Courcelle's Theorem . . .

. . . rephrased in the parlance of parameterized complexity:

Theorem (Courcelle, 1990)

Let $\varphi \in MSO_2$ and let $\mathcal{C}$ be the class of all graphs. Then MSO_2 model-checking problem $MC(MSO_2, \mathcal{C})$: "Does $G \models \varphi$?" is fixed-parameter tractable wrt the parameter $|\varphi| + tw(G)$.

Extended to **(directed) graphs with vertex/edge labels** (from a finite set) and **problems involving evaluations of sets** definable in MSO [Arnborg, Lagergren and Seese, 1991].

No **lower bounds** were known till recently.

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Courcelle's Theorem: Lower Bounds

Are there classes of **unbounded treewidth** for which Courcelle's Theorem holds?

YES!

Let $\mathcal{C} = \{G \mid \mathrm{tw}(G) = \log^* |G|\}$. Given an MSO-formula φ and an n-vertex graph $G \in \mathcal{C}$, time taken to decide $G \models \varphi$:

$$\exp^{(|\varphi|)}(\mathrm{tw}(G)) \cdot n \leq \exp^{(|\varphi|)}(\mathrm{tw}(G)) \cdot \exp^{(\log^* n)}(\log^* n) \leq n^2,$$

where $\exp^{(0)}(x) = x$ and

$$\exp^{(i)}(x) = 2^{\exp^{(i-1)}(x)}.$$

Question

How fast must the treewidth grow for Courcelle's Theorem to fail?

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Courcelle's Theorem: Lower Bounds . . .

Theorem (Makowsky and Mariño, 2004)

If $\mathcal{C}$ is a class of graphs of unbounded treewidth that is closed under topological minors and $G \in \mathcal{C}$, then deciding whether $G \models \varphi$ is not in FPT wrt $|\varphi|$ as parameter unless $P = NP$.

- Closure under topological minors is a very strong restriction.
- Kreutzer and Tazari: Similar result without this restriction for graph classes with **moderately unbounded treewidth**.

Classes of Unbounded Treewidth

Definition (Bounded Treewidth)

Let $f : \mathbf{N} \to \mathbf{N}$. A class $\mathcal{C}$ of graphs have f-bounded-treewidth if for all $G \in \mathcal{C}$, we have that $tw(G) \leq f(|G|)$.

Examples

- Courcelle's Theorem: $f(n) := c$, a constant.
- $f(n) := n$ is the maximum function that makes sense.
- In Kreutzer and Tazari: $f(n) := \log^c n$, for some constant $c > 0$.

Polylogarithmically Unbounded Classes

Definition (Kreutzer and Tazari)

The treewidth of a graph class $\mathcal{C}$ is polylogarithmically unbounded if for all $c > 1$ the following holds: for all $n \in \mathbf{N}$ there exists $G_n \in \mathcal{C}$ with

- $\log^c(|G_n|) \leq tw(G_n)$ *(unboundedness);*
- $n \leq tw(G_n) \leq n^\gamma$, *for some fixed γ (density);*
- G_n *can be constructed in time 2^{n^ϵ}, for some fixed $\epsilon < 1$ (constructibility).*

Note

$$\log^c(|G_n|) \leq \text{tw}(G_n) \leq n^\gamma \implies |G_n| \leq 2^{n^{\gamma/c}}.$$

Courcelle's Theorem: A Lower Bound

Theorem (Kreutzer and Tazari, 2010)

Let $\mathcal{C}$ be a graph class with the following properties:

- *$\mathcal{C}$ is closed under subgraphs;*
- *the treewidth of $\mathcal{C}$ is polylogarithmically unbounded.*

Then $MC(MSO_2, \mathcal{C})$ is not in XP ($|G|^{f(|\varphi|)}$ for any computable f), unless SAT can be solved in subexponential time.

High-level Proof Idea

Reduce Sat to $\mathrm{MC}(\mathrm{MSO}_2, \mathcal{C})$.

- *Input:* A SAT formula F of length n.
- *Question:* Is F satisfiable?

Reduction

1. Construct $G_n \in \mathcal{C}$ of treewidth n^d s.t. $\log^c(|G_n|) < \mathrm{tw}(G_n)$ and $c > d$.
   - Conditions 1 and 2: G_n exists in $\mathcal{C}$.
   - Condition 3: G_n is efficiently constructible and $|G_n| < 2^{n^{d/c}}$.
2. Encode F in a subgraph of G_n (exists because $\mathrm{tw}(G_n) \approx n^d$).
   - Using closure under subgraphs.
3. Define an MSO-formula φ (independent of F) s.t. F satisfiable iff $G_n \models \varphi$.
   - Deciding $G_n \models \varphi$ in XP takes time $2^{n^{c/d} \cdot f(|\varphi|)}$, subexponential in $|F|$.

A Critique of Kreutzer & Tazari's Result

- There are classes $\mathcal{C}$ closed under subgraphs with logarithmic treewidth s.t. $\mathrm{MC}(\mathrm{MSO}_2, \mathcal{C})$ is in XP [Makowski and Mariño, 2004].
  - Threshold for treewidth is more-or-less strict.
- The constructibility clause in the definition of polylogarithmically unbounded treewidth is unnatural.
- Proofs are very technical and spread over several papers.

Outline

Main Theorem I

Theorem

Let $\mathcal{C}$ be a graph class s.t.

- *$\mathcal{C}$ is closed under subgraphs;*
- *the treewidth of $\mathcal{C}$ is polylogarithmically unbounded.*

Then the MSO_1 model-checking problem on vertex labeled graphs from $\mathcal{C}$ is not in XP, unless 3-Colorability is in time $2^{o(n)}$ with subexponential advice.

- The labels are from a fixed, finite set.
- Nonuniform ETH: SAT, 3-Colorability are not in $2^{o(n)}$ time with subexponential advice.

Major Differences Between the Two Results

1. We use a weaker logic.
   - Our result: applies to MSO_1 model-checking on vertex-labeled graphs.
   - K & T's result: applies to MSO_2 model-checking on unlabeled graphs.
2. No constructibility requirement.
   - We use a stronger complexity assumption: Nonuniform ETH.
3. Easy proofs!

MSO_2 versus MSO_1 with Vertex Labels

MSO_1 with vertex labels is weaker than MSO_2.

- Hamiltonian Path/Cycle cannot be expressed in MSO_1 with vertex labels.

Results such as Courcelle's Theorem and Courcelle, Makowski and Rotics's Theorem for rankwidth can be extended to vertex-labeled graphs.

- Extending C,M,R's Theorem for rankwidth from MSO_1 to MSO_2 would imply **EXP = NEXP**.

On the Constructibility Clause

Our definition of polylogarithmically unbounded treewidth:

Definition

The treewidth of a graph class $\mathcal{C}$ is polylogarithmically unbounded if there is a constant γ s.t. for all $c > 1$ the following holds. For all $n \in \mathbf{N}$ there exists $G_n \in \mathcal{C}$ with

- $\log^c(|G_n|) \leq tw(G_n)$ *(unboundedness);*
- $n \leq tw(G_n) \leq n^\gamma$ *(density).*

Note: $|G_n| \leq 2^{n^{\gamma/c}}$.

- No constructibility requirement.
- At the expense of a stronger complexity-theoretic assumption: Nonuniform ETH.

ETH versus Nonuniform ETH (NETH)

Exponential Time Hypothesis [Impagliazzo, Paturi, and Zane, 2001]:

- n-variable 3-SAT cannot be solved in $2^{o(n)}$ time.
- Can be formulated using other problems such as Vertex Cover or 3-Colorability.

NETH: n-variable 3-SAT not solvable in $2^{o(n)}$ time using:

- a family of algorithms, one for each input length;
- a circuit-family $\mathcal{F}$ s.t. for each input length n, $\exists C_n \in \mathcal{F}$ with $|C_n| \leq 2^{o(n)}$;
- an algorithm that receives oracle advice which depends only on the input length n and has $2^{o(n)}$ bits.

Can be formulated in terms of Vertex Cover or 3-Colorability.

Main Theorem II

Our result can be strengthened by assuming that the label set is arbitrary but finite.

Theorem

Let L be a finite label set and let $\varphi \in MSO_1[L]$. Let $\mathcal{C}$ be a graph class s.t.

- *$\mathcal{C}$ is closed under subgraphs;*
- *the treewidth of $\mathcal{C}$ is polylogarithmically unbounded.*

Then the MSO_1 model-checking problem on vertex labeled graphs from $\mathcal{C}$ is not in XP, unless all problems in PH can be solved in time $2^{o(n)}$ with subexponential advice.

Outline

Main Theorem I

Theorem

Let $\mathcal{C}$ be a graph class s.t.

- *$\mathcal{C}$ is closed under subgraphs;*
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Then the MSO_1 model-checking problem on vertex labeled graphs from $\mathcal{C}$ is not in XP, unless 3-Colorability is in time $2^{o(n)}$ with subexponential advice.

Proof. A multistage reduction from 3-Colorability.

Proof Idea: Stage I

Let $\varphi' \in \text{MSO}_1$ express 3-Colorability and let H' be an instance of this problem.

Reduce $(H', \varphi') \to (H, \varphi)$ in polynomial time s.t.

- H is $\{1, 3\}$-planar;
- φ depends only on φ' and $|\varphi| = O(|\varphi'|)$.
- $H' \models \varphi'$ iff $H_{sub} \models \varphi$ for every subdivision H_{sub} of H.

Note that

- φ is an "interpretation" of 3-Colorability closed under edge subdivisions;
- $|H'| = n$ and $|H| \leq n^b$ for some constant b.

Proof Idea: Grid-Like Subgraphs

Polylogarithmic Unboundedness of tw $(\mathcal{C})$

- $\exists G_n \in \mathcal{C}$ s.t. $\mathrm{tw}(G_n) \geq \log^c(|G_n|)$ and $n^{5b} \leq \mathrm{tw}(G_n) \leq n^{5b\gamma}$.
- $|G_n| \leq 2^{n^{5b\gamma/c}}$ for $c > 5b\gamma$.

Grid-Like Subgraphs [Reed and Wood, 2008]

- $\mathrm{tw}(G_n) \geq n^{5b}$ implies G_n contains a **grid-like subgraph Γ_{n^b} of order n^b**.
- Γ_{n^b} "contains" a subdivision H_{sub} of H.

Closure of $\mathcal{C}$ under Subgraphs

- $\Gamma_{n^b} \in \mathcal{C}$.

Proof Idea: Stage II

Lemma

Let Γ_{n^b} "contain" K and let $\varphi \in MSO_1$. There is a fixed finite set L s.t. one can in poly time construct a labeling $\lambda : V(\Gamma_{n^b}) \to L$ and $\psi \in MSO_1[L]$ (depends only on φ) s.t.

$$K \models \varphi \text{ iff } (\Gamma_{n^b}, \lambda) \models \psi.$$

- Since Γ_{n^b} "contains" H_{sub}, we have:

$$H' \models \varphi' \text{ iff } H \models \varphi \text{ iff } H_{sub} \models \varphi \text{ iff } (\Gamma_{n^b}, \lambda) \models \psi.$$

- $|\Gamma_{n^b}| \leq 2^{n^{5b/c}}$; supplied as advice of subexponential size.

Time taken to decide $H' \models \varphi'$ is $|\Gamma_{n^b}|^{f(|\psi|)} = 2^{o(n)}$.

Outline

Consequences for Directed Width Measures

Extension of [Ganian et al., 2010].

Theorem

Unless NETH fails, there exists no directed width measure δ satisfying following three properties:

1. δ *is closed under subdigraphs;*
2. $\exists$ *digraph class* $\mathcal{C}$ *of bounded* δ*-width with* $tw(\mathcal{C})$ *polylogarithmically unbounded;*
3. *for L-vertex-labeled digraphs D and* $\varphi \in MSO_1[L]$*, deciding* $D \models \varphi$ *is in time* $O(|D|^{f(\delta(D),|\varphi|)})$.

Summary

Main Contribution

- Strengthen and simplify Kreutzer and Tazari's impressive result.

Extending to Unlabeled MSO_1?

- **Open.** Is there is a (nontrivial) graph class where model-checking MSO_1 is easy but $MSO_1[L]$ is hard?
- This indicates that the result might be extendable to unlabeled MSO_1.

Thank You!