# The Complexity of Finding Subgraphs Whose Matching Number Equals the Vertex Cover Number

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**Abstract.** The class of graphs where the size of a minimum vertex cover equals that of a maximum matching is known as König-Egerváry graphs. König-Egerváry graphs have been studied extensively from a graph theoretic point of view. In this paper, we introduce and study the algorithmic complexity of finding maximum König-Egerváry subgraphs of a given graph. More specifically, we look at the problem of finding a minimum number of vertices or edges to delete to make the resulting graph König-Egerváry. We show that both these versions are NP-complete and study their complexity from the points of view of approximation and parameterized complexity. En route, we point out an interesting connection between the vertex deletion version and the Above Guarantee Vertex Cover problem where one is interested in the parameterized complexity of the Vertex Cover problem when parameterized by the 'additional number of vertices' needed beyond the matching size. This connection is of independent interest and could be useful in establishing the parameterized complexity of Above Guarantee Vertex Cover problem.

## 1 Introduction

One of the celebrated min-max results of graph theory is the König-Egerváry theorem which states that for bipartite graphs the size of a minimum vertex cover equals that of a maximum matching. The class of graphs for which equality holds includes bipartite graphs as a proper subclass and is known as König-Egerváry graphs. König-Egerváry graphs have been studied extensively in the literature from a graph theoretic point of view. A good characterization of König-Egerváry graphs was found independently by Deming [4] and Sterboul [16]. Recently in [7], Korach, Nguyen and Peis have presented an excluded subgraph characterization of König-Egerváry graphs. In this paper, we define various problems related to finding König-Egerváry subgraphs and study their algorithmic complexity from the points of view of parameterized complexity and approximation algorithms. More precisely the problems which we study in this paper are:

- König Vertex (Edge) Deletion Set (KVDS (KEDS)): Given a graph $G = (V, E)$ and a positive integer $k$, does there exist $V' \subseteq V$ ($E' \subseteq E$) such that $|V'|$ ($|E'|$) $\leq k$ and $G[V - V']$ ($G' = (V, E - E')$) is a König-Egerváry graph?
- Maximum Vertex (Edge) Induced König Subgraph (MVIKS (MEIKS)): Given a graph $G = (V, E)$ and a positive integer $k$, does there exist $V' \subseteq V$ ($E' \subseteq E$) such that $|V'|$ ($|E'|$) $\geq k$ and $G[V']$ ($G' = (V, E')$) is a König-Egerváry graph?

In order to explain one motivation for studying the König-Egerváry subgraph problem, we need to digress and discuss about parameterized complexity. In the framework of parameterized complexity, one deals with decision problems whose inputs consist of a pair $(x, k)$, where $k$ is called the parameter; the goal is to decide whether $(x, k)$ is a yes-instance or not in time $O(f(k) \cdot |x|^{O(1)})$, where $f$ is a function of $k$ alone. Decision problems that admit such algorithms are called *fixed-parameter tractable* (FPT). For more on parameterized complexity see the recent textbook by Niedermeier [12]. Over the last decade or so a number of *NP*-hard problems have been shown to be fixed-parameter tractable. One of the most famous fixed-parameter tractable problems is Vertex Cover. An input to this problem is a graph $G = (V, E)$ and a positive integer $k$ and the goal is to decide whether there exists a set of at most $k$ vertices which covers all edges. Over the years a lot of work has been done to devise better FPT algorithms for this problem; the current best algorithm runs in time $O(1.2738^k + kn)$, where $n = |V|$ [3].

Since the size of a maximum matching size is a lower bound for vertex cover, a natural generalization of the Vertex Cover problem is the following:

> Above Guarantee Vertex Cover ($g$-VC): Let $G = (V, E)$ be a graph with maximum matching size $\mu(G)$ and $k$ a positive integer. The goal is to decide whether $G$ admits a vertex cover of size at most $\mu(G) + k$, where $k$ is the parameter.

This problem (and in general, such above guarantee problems, see [10]) seems difficult. In this paper, we show that this problem is fixed-parameter equivalent to KVDS. As a corollary, we obtain an $O(n^k)$ algorithm for $g$-VC. To the best of our knowledge, even this was not known before.

Our second reason for studying König-Egerváry subgraph problems is that the versions of König-Egerváry subgraph problems when the resulting graph we look for is bipartite (i.e. replace König-Egerváry in the above problem definitions by *bipartite*) are well studied in the area of approximation algorithms and parameterized complexity [9, 13, 14]. König-Egerváry subgraph problems are natural generalizations of bipartite subgraph problems but have not been studied algorithmically. We believe that this can trigger explorations of other questions in König-Egerváry graphs. For the rest of the paper, we use König as an abbreviation of König-Egerváry.

The remaining paper is organized as follows. In Section 2, we state some results and notation that we make use of in the rest of the paper. In Section 3, we consider vertex versions and in Section 4, the edge versions of the König Subgraph problem. We study both versions from the points of view of approximation and parameterized complexity. We conclude in Section 5 with a list of open problems.

## 2 Preliminaries

We will make use of the following well-known results in the rest of the paper.

**Lemma 1.** [7] *If $G = (V, E)$ is König then there exists a bipartition of $V = V_1 \uplus V_2$ such that $V_2$ is independent and there exists a matching that saturates $V_1$ and crosses the cut $(V_1, V_2)$.*

**Lemma 2.** [4] *Let $G$ be a graph on $n$ vertices and $m$ edges. One can check whether $G$ is König and if it is König find a minimum vertex cover of $G$ in time $O(m\sqrt{n})$.*

Deming [4] gave an $O(n+m)$ algorithm to check whether a graph $G$ is König provided a maximum matching of $G$ is given as part of the input. Since computing a maximum matching takes time $O(m\sqrt{n})$ [11], testing whether a graph is König takes time $O(m\sqrt{n})$.

We use $\mu(G)$ and $\beta(G)$ to denote, respectively, the size of a maximum matching and minimum vertex cover of $G$. When the graph being referred to is clear from the context, we simply use $\mu$ and $\beta$. We sometimes use $\tau(G)$ to denote the difference $\beta(G) - \mu(G)$. Unless otherwise stated, we will use $n$ and $m$ to denote, respectively, the number of vertices and the number of edges of a graph. All graphs in this paper are simple and undirected.

## 3 Vertex Versions of the König Subgraph Problems

The problems we consider in this section are KVDS and MVIKS. König will be denoted by $\kappa(G)$. Since König graphs are a "generalization" of bipartite graphs and since the Vertex Bipartization problem is now known to be FPT [13], the parameterized complexity of the KVDS problem is very interesting.

In the next subsection, we relate the KVDS problem with another important open problem in the area of parameterized complexity, the $g$-VC. This also proves the NP-completeness of the KVDS problem.

### 3.1 König Vertex Deletion Set and Above Guarantee Vertex Cover

We begin with a result which states essentially that for the $g$-VC problem we may, without loss of generality, assume that the input graph to have a perfect matching.

Let $G = (V, E)$ be an undirected graph and let $M$ be a maximum matching of $G$. Construct $G' = (V', E')$ as follows. $G'$ has all the vertices and edges that $G$ has. Let $I$ be the independent set with respect to the matching $M$, that is, $I = V - V[M]$. Let $V' = V \cup \{u' : u \in I\}$ and $E' = E \cup \{\{u', v\} : \{u, v\} \in E\} \cup \{\{u, u'\} : u \in I\}$. Then $M' = M \cup \{\{u, u'\} : u \in I\}$ is a perfect matching for $G'$. Then we have

**Theorem 1.** *Let $G$ be a graph without a perfect matching and let $G'$ be the graph obtained by the above construction. $G$ has a vertex cover of size $\mu(G) + k$ if and only if $G'$ has a vertex cover of size $\mu(G') + k$.*

*Proof.* Let $M$ denote a maximum matching of $G$, $I$ denote the set $V(G) \setminus V[M]$ and $I'$ denote the new set of vertices that are added in constructing $G'$. Clearly, $\mu(G') = \mu(G) + |I|$.

($\Rightarrow$) Let $C$ be a vertex cover of $G$ of size $\mu(G) + k$. Define $C_I = C \cap I$ and $C' = C \cup (I \setminus C_I) \cup \{u' \in I' : u \in C_I \text{ and } \{u, u'\} \in E(G')\}$. We claim that $C'$ is a vertex cover of $G'$ of size $\mu(G') + k$. $C'$ clearly covers all edges of $G$ as well as edges of the form $\{u, u'\}$, where $u \in I$ and $u' \in I'$. Moreover if $\{u', w\}$ is an edge of $G'$ where $u' \in I'$ and $w \in V[M]$, then $\{u, w\}$ is an edge of $G$. Therefore either $w \in C$ in which case $C'$ covers the edge $\{u', w\}$; or, $u \in C$ in which case $u' \in C'$. Thus $C'$ is a vertex cover of $G'$. Also $|C'| = |C| + |I \setminus C_I| + |C_I| = |C| + |I| = \mu(G) + k + |I| = \mu(G') + k$.

($\Leftarrow$) Let $C'$ be a vertex cover of $G'$ of size $\mu(G') + k$. Define $M'$ to be the set of edges of the form $\{\{u, u'\} : u \in I \text{ and } u' \in I'\}$ such that both endpoints are in $C'$. One can show that $C = (C' \cap V[M]) \cup \{u \in I : \{u, u'\} \in M'\}$ is a vertex cover of $G$ of size $\mu(G) + k$. ∎

The next theorem relates the vertex cover of a graph with the König vertex deletion set.

**Theorem 2.** *Let $G$ be an $n$-vertex graph with a perfect matching. $G$ has a vertex cover of size at most $\frac{n}{2} + k$ if and only if $G$ has a König vertex deletion set of size at most $2k$.*

*Proof.* ($\Rightarrow$) Let $P$ be a perfect matching of $G$ and $C$ a vertex cover of $G$ of size at most $\frac{n}{2} + k$. Consider the subset $M \subseteq P$ of matching edges both of whose endpoints are in $C$. Clearly $V[M]$ is a König vertex deletion set of $G$ of size at most $2k$.

($\Leftarrow$) Conversely let $K$ be a König vertex deletion set of $G$ of size $r \leq 2k$. Then $G - K$ is a König graph on $n - r$ vertices and hence has a vertex cover $C'$ of size at most $\frac{n-r}{2}$. Clearly $C = C' \cup K$ is a vertex cover of $G$ of size $|C'| + |K| \leq \frac{n-r}{2} + r = \frac{n+r}{2} \leq \frac{n}{2} + k$. ∎

The following corollary follows from Theorems 1 and 2 and the fact that Vertex Cover is NP-complete.

**Corollary 1.** *The* König Vertex Deletion Set *problem is NP-complete.*

**Corollary 2.** *Let $G$ be an $n$-vertex graph with a perfect matching $P$. $G$ has a minimum vertex cover of size $\frac{n}{2} + k$ if and only if $G$ has a minimum König vertex deletion set of size $2k$. Moreover there exists an edge subset $M \subseteq P$ of size $k$ such that $V[M]$ is a minimum König vertex deletion set of $G$.*

If we let $\tau(G) = \beta(G) - \mu(G)$, then the above corollary states: $\kappa(G) = 2\tau(G)$.

We have shown that for graphs with a perfect matching, the KVDS problem is fixed-parameter equivalent to the $g$-VC problem. It is not obvious how to check whether a graph $G$ has a vertex cover of size $\mu(G) + k$ in time $O^*(n^k)$[3]. The following theorem shows how this may be done.

**Theorem 3.** *Let $G$ be a graph on $n$ vertices and $m$ edges. One can determine whether $G$ has a vertex cover of size at most $\mu(G) + k$ in time $O(m\sqrt{n} + \binom{n}{k}(m + n))$.*

*Proof.* If $G$ does not have a perfect matching, construct $G'$ as in Theorem 1 in linear time. $G'$ has a perfect matching and at most $2n$ vertices and $2m$ edges. By Theorems 1 and 2, $G$ has a vertex cover of size at most $\mu(G)+k$ if and only if $G'$ has a König vertex deletion set of size at most $2k$. By Corollary 2, to check whether $\kappa(G') \leq 2k$, all we need to do is select subsets of $k$ edges of a perfect matching of $G'$ and for each edge set, delete the endpoints of the edges and check whether the remaining graph is König. Obtaining a perfect matching takes time $O(m\sqrt{n})$. Cycling through all possible edge sets and testing whether the remaining graph is König takes time $O(\binom{n}{k}(m + n))$. Hence the claim. ■

For a graph $G$ with a perfect matching, Theorem 2 relates the size of a vertex cover of $G$ with that of a König vertex deletion set of $G$. For graphs without a perfect matching, we have following result.

**Theorem 4.** *Let $G$ be a graph without a perfect matching. If $G$ has a vertex cover of size $\mu(G)+k$ then $G$ has a König vertex deletion set of size at most $2k$. Moreover, $\tau(G) \leq \kappa(G) \leq 2\tau(G)$, where $\tau(G) = \beta(G) - \mu(G)$.*

*Proof.* Let $M$ be a maximum matching of $G$ and let $C$ be a vertex cover of $G$ of size $\mu(G) + k$. Define $I = V \setminus V[M]$, $C_I = C \cap I$ and $M'$ to be the subset of $M$ both of whose endpoints are in $C$. One can then verify that $V[M'] \cup C_I$ is a König cover of $G$ of size at most $2k$.

This shows that $\kappa(G) \leq 2\tau(G)$. To prove that $\tau(G) \leq \kappa(G)$, suppose that there exists $S \subseteq V$, $|S| < \tau(G)$, such that $G \setminus S$ is König. Then the following easily verifiable inequalities

$$\mu(G \setminus S) \leq \mu(G)$$
$$\beta(G \setminus S) \geq \beta(G) - |S| = \mu(G) + \tau(G) - |S|$$

imply that $\beta(G \setminus S) > \mu(G \setminus S)$, a contradiction. ■

### 3.2 Approximability Results

**An $O(\log n \log\log n)$-Approximation Algorithm:** Chen and Kanj [2] obtain an approximation for the Vertex Cover problem on graphs with a perfect matching by reducing Vertex Cover on graphs with a perfect matching to Max 2-Sat and then using an approximation algorithm for Max 2-Sat. Observe that by Theorem 1, we may assume without loss of generality that the input graph has a perfect matching. We observe that using the approximation algorithm for Min 2-Sat gives a good approximation for $\tau(G)$ and hence for $\kappa(G)$ and thus for graphs whose vertex cover size differs by a small amount the maximum matching size.

[3] The $O^*$ notation suppresses polynomial terms.

We now describe the reduction to Min Weight 2-Sat Del. Recall that an instance of Min Weight 2-Sat Del is a 2-CNF formula whose clauses have weights associated with them and the question is to delete clauses of minimum total weight so that the resulting formula is satisfiable. This problem is NP-complete but has an $O(\log n \log\log n)$-approximation algorithm [6]. Let $G = (V, E)$ be a graph with a perfect matching $P$. For every vertex $u \in V$, define $x_u$ to be a Boolean variable. Let $\mathcal{F}(G, P)$ denote the Boolean formula

$$\mathcal{F}(G,P) = \bigwedge_{(u,v)\in P} (\bar{x}_u \vee \bar{x}_v) \bigwedge_{(u,v)\in E} (x_u \vee x_v).$$

The proof of the next lemma follows from Theorem 2 and the proof of Theorem 5.1 in [2].

**Lemma 3.** *Let $G = (V, E)$ be an $n$-vertex graph with a perfect matching $P$. Then the following three statements are equivalent:*

1. *$G$ has a vertex cover of size $\frac{n}{2} + k$.*
2. *$G$ has a König vertex deletion set of size $2k$.*
3. *There exists an assignment that satisfies all but at most $k$ clauses of $\mathcal{F}(G, P)$.*

From the proof of Theorem 5.1 in [2], it follows that if there exists an assignment that satisfies all but at most $k$ clauses of $\mathcal{F}(G, P)$ then we may, without loss of generality, assume these to be of the form $(\bar{x}_u \vee \bar{x}_v)$, where $(u, v) \in P$, that is, the clauses that correspond to the perfect matching. This is important to our approximation algorithm that we are about to present.

We need the following result on the Min Weight 2-Sat Del problem for our approximation algorithm.

**Lemma 4.** [1, 6] *Let $\Phi$ be an instance of* Min Weight 2-Sat Del *with $n$ variables. One can in polynomial time obtain a solution that has weight $O(\log n \log\log n)$ times the optimal. If we are willing to allow randomness, we can obtain a solution that has weight $O(\sqrt{\log n})$ times the optimal.*

Our approximation algorithm for the $g$-VC and KVDS problems is presented in Figure 1. Given an $n$-vertex graph $G$ with a maximum matching of size $\mu$ and a minimum vertex cover of size $\beta$, this algorithm outputs a vertex cover of $G$ of size at most $\mu + O(\log n \log\log n)(\beta - \mu)$. Thus this algorithm approximates the deficit between the sizes of a minimum vertex cover and a maximum matching. There exists a 2-approximation algorithm for the Vertex Cover problem which simply includes all vertices of a maximum matching. It is an interesting open problem to devise a polynomial time algorithm which has an approximation factor less than 2.

Our algorithm is better than any *constant* factor approximation algorithm for Vertex Cover whenever $\beta - \mu = o(\frac{n}{\log n \log\log n})$. To see this, note that a $c$-approximate algorithm, $c > 1$, outputs a solution of size $\mu c + (\beta - \mu)c$ whereas our algorithm outputs a solution of size $\mu + \alpha(\beta - \mu)$, where $\alpha = O(\log n \log\log n)$. Now if $\beta - \mu = o(\frac{\mu}{\alpha})$, then $\mu + \alpha(\beta - \mu) < \mu c + (\beta - \mu)c$. But $\mu = O(n)$ and hence if $\beta - \mu = o(\frac{n}{\log n \log\log n})$ our algorithm performs better than the $c$-approximate algorithm for any $c > 1$.

From Lemmas 3 and 4, we get

**Theorem 5.** *Let $G$ be a graph on $n$ vertices with a maximum matching of size $\mu$ and a minimum vertex cover of size $\beta$. Then* Algo-Above-Guar-Vertex-Cover *finds a vertex cover of $G$ of size $\mu + O(\log n \log\log n)(\beta - \mu)$.*

Note that $V(\mathcal{S})$ is actually a König vertex deletion set of $G$. Since $|V(\mathcal{S})| \leq O(\log n \log\log n)\ (\beta - \mu)$, we have

**Theorem 6.** *Given a graph $G$ on $n$ vertices, there exists an algorithm that approximates the König vertex deletion set of $G$ to within a factor of $O(\log n \log\log n)$.*

One can obtain a randomized algorithm for the $g$-VC and KVDS problems using the $O(\sqrt{\log n})$-randomized approximation algorithm for Min Weight 2-Sat Del, mentioned in Lemma 4, in Step 3 of the algorithm.

| Algo-Above-Guar-Vertex-Cover |
| --- |
| **Step 1.** If $G$ does not have a perfect matching, construct $G'$ as in Theorem 1 and set $H \leftarrow G'$; else set $H \leftarrow G$. |
| **Step 2.** Find a maximum matching $M$ of $H$ and construct $\mathcal{F}(H, M)$. |
| **Step 3.** Use the approximation algorithm for Min Weight 2-Sat Del to obtain an $O(\log n \log\log n)$-approximate solution $\mathcal{S}$ for $\mathcal{F}(H, M)$, where $n = \lvert V(H) \rvert$. |
| **Step 4.** Obtain a minimum vertex cover $C$ of the König graph $H - V(\mathcal{S})$, where $V(\mathcal{S})$ is the set of vertices corresponding to $\mathcal{S}$. |
| **Step 5.** If $H = G$ return $C \cup V(\mathcal{S})$; else return $(C \cup V(\mathcal{S})) - (V(G') - V(G))$. |

**Fig. 1.** Approximation Algorithm for Above Guarantee Vertex Cover

**A Hardness Result:** In this subsection we show the hardness of approximating the MVIKS problem. We show this by a reduction from the Independent Set problem to the MVIKS.

**Theorem 7.** *There is no approximation algorithm for MVIKS with factor* $O(n^{1-\epsilon})$*, for any* $\epsilon > 0$*, unless NP-complete problems have randomized approximation algorithms.*

*Proof.* We give a reduction from Independent Set to the MVIKS problem. Given an instance $(G, k)$ of Independent Set, construct a graph $H$ as follows. The vertex set of $H$ consists of two copies of $V(G)$ namely, $V_1 = \{u_1 : u \in V(G)\}$ and $V_2 = \{u_2 : u \in V(G)\}$. For all $u \in V(G)$, $(u_1, u_2) \in E(H)$. If $(u, v) \in E(G)$, add the edges $(u_1, v_1)$, $(u_2, v_2)$, $(u_1, v_2)$ and $(v_1, u_2)$ in $E(H)$. $H$ has no more edges.

We claim that $G$ has an independent set of size $k$ if and only if $H$ has a König subgraph of size $2k$. Let $I$ be an independent set of size $k$ in $G$. Let $K = \{u_1, u_2 \in V(H) : u \in I\}$. Clearly $G[K]$ is an induced matching on $2k$ vertices and hence König. Conversely, let $K$ be a König subgraph of $H$ on $2k$ vertices. Since every König graph on $n$ vertices has an independent set of size at least $n/2$, let $I'$ be an independent set of $K$ of size at least $k$. Define $I = \{u \in V(G) : \text{either } u_1 \text{ or } u_2 \in I'\}$. It is clear that the vertices of $I'$ correspond to distinct vertices of $G$ and hence $|I| \geq k$. It is also easy to see that the vertices in $I$ actually form an independent set. Since the Independent Set problem has no approximation algorithms with factor $O(n^{1-\epsilon})$, for any $\epsilon > 0$, unless *NP*-complete problems have randomized approximation algorithms, this completes the proof. ■

In the reduction above, $|V(H)| = 2|V(G)|$ and $(G, k)$ is a yes-instance of Independent Set if and only if $(H, 2k)$ is a yes-instance of MVIKS problem. Thus this reduction can also be viewed as a reduction from Vertex Cover to KVDS problem. Dinur and Safra [5] have shown that unless P = NP, the Vertex Cover problem cannot be approximated to within 1.3606. Using this, one can easily show that the KVDS problem cannot be approximated to within 1.3606 unless P ≠ NP. The result of Dinur and Safra [5] also holds for graphs with a perfect matching. Using this fact, we show a stronger hardness result for the KVDS problem.

**Corollary 3.** *Under the hypothesis P ≠ NP, the KVDS problem cannot be approximated to within* 1.7212*.*

*Proof.* It is sufficient to prove this result for graphs with a perfect matching. Let $\mathcal{A}$ be a $d$-approximation algorithm for the König Cover problem in graphs with a perfect matching. Let $G$ be an $n$-vertex graph with a perfect matching. Using $\mathcal{A}$, one can obtain a König cover of size at most $d\kappa(G)$ and hence a vertex cover of size at most $\frac{n - d\kappa(G)}{2} + d\kappa(G) = \frac{n + d\kappa(G)}{2}$. An optimum vertex cover of $G$ has size $\frac{n+\kappa(G)}{2}$. By Dinur and Safra [5], we must have $\frac{n + d\kappa(G)}{n + \kappa(G)} \geq 1.3606$. Simplifying this yields $\frac{n}{\kappa(G)} \leq \frac{d - 1.3606}{0.3606}$. Note that $n \geq \kappa(G)$ and so $d \geq 1.7212$. ■

Also note that the reduction in Theorem 7 is a fixed parameter reduction and thus the parameterized version of MVIKS problem is $W[1]$ hard.

**Corollary 4.** *The MVIKS problem is* $W[1]$*-hard.*

## 4 Edge Versions of the König Subgraph Problem

In this section we look at edge version of König Subgraph problem.

### 4.1 NP-Completeness

We show that MEIKS problem is NP-complete by reducing the NP-complete Min 2-Sat Deletion problem to the KEDS problem.

**Theorem 8.** *The* König Edge Deletion Set (KEDS) *problem is NP-complete.*

*Proof.* We give a reduction from Min 2-Sat Deletion. Let $\Phi = \bigwedge_{i=1}^{m}(y_i \vee z_i)$ be a 2-Sat formula. Construct a graph $G_\Phi = (V, E)$ as follows. Suppose the formula $\Phi$ is composed of the literals $\{x_1, \bar{x}_1, \ldots, x_n, \bar{x}_n\}$. The vertex set $V$ of $G_\Phi$ is defined as follows:

$$V = \{x_1, \bar{x}_1, \ldots, x_n, \bar{x}_n, x_{11}, \bar{x}_{11}, \ldots, x_{1,k+1}, \bar{x}_{1,k+1}, \ldots, x_{n,1}, \bar{x}_{n,1}, \ldots, x_{n,k+1}, \bar{x}_{n,k+1}\}.$$

The edge set $E = E_1 \cup E_2 \cup E_3 \cup E_4$, where $E_1, E_2, E_3, E_4$ are defined below.

$$\begin{aligned} E_1 &= \{(x_i, \bar{x}_i) : 1 \le i \le n\} \\ E_2 &= \{(x_{i,j}, \bar{x}_{i,l}) : 1 \le j, l \le k+1, 1 \le i \le n\} \\ E_3 &= \{(x_i, \bar{x}_{i,j}), (\bar{x}_i, x_{i,j}) : 1 \le i \le n, 1 \le j \le k+1\} \\ E_4 &= \{(y_i, y_j) : (y_i \vee y_j) \in \Phi\} \end{aligned}$$

Note that $G_\Phi$ has a perfect matching and that each clause of $\Phi$ corresponds to an edge of $G_\Phi$ (the edges in $E_4$). Think of the literals $x_{i,1}, \ldots, x_{i,k+1}$ as copies of the literal $x_i$. In any assignment to the variables of $\Phi$, think of the copies getting the same value as the original literal.

*Claim.* There exists an assignment satisfying all but $k$ clauses of $\Phi$ if and only if there exist at most $k$ edges whose deletion makes $G_\Phi$ König.

($\Rightarrow$) Let $\alpha$ be an assignment to the variables of $\Phi$ that satisfies all but $k$ clauses. Each of these $k$ clauses corresponds to a distinct edge in $G_\Phi$. Delete these edges from $G_\Phi$. Then for each edge $e$ in the remaining graph, at least one endpoint of $e$ is assigned 1 by $\alpha$. To prove that the remaining graph is König, we must demonstrate a bipartition of the vertex set into $V_1 \uplus V_2$ (say) such that $V_2$ is independent and there exists a matching saturating $V_1$ which crosses the cut $(V_1, V_2)$. If $\alpha(x_i) = 1$ then place the vertices $x_i, x_{i,1}, \ldots x_{i,k+1}$ in $V_1$; else place $\bar{x}_i, \bar{x}_{i,1}, \ldots \bar{x}_{i,k+1}$ in $V_1$. The remaining vertices are placed in $V_2$. Clearly $V_2$ is independent. Note that if $x_i \in V_1$ then $\bar{x}_i \in V_2$ and vice versa. Also if $x_{i,j} \in V_1$ then $\bar{x}_{i,j} \in V_2$ and vice versa. Hence there exists a matching that saturates $V_1$ and crosses the cut $(V_1, V_2)$.

($\Leftarrow$) Conversely suppose that deleting $k$ edges makes $G_\Phi$ König. We will assume that this set of edges is a minimal edge deletion set. Call the resulting graph $G'_\Phi$. Then the vertex set of $G'_\Phi$ can be partitioned into $V_1$ and $V_2$ such that $V_2$ is independent and there exists a matching saturating $V_1$ that crosses the cut $(V_1, V_2)$.

*Claim.* For each $1 \le i \le n$, it is not the case that $x_i, \bar{x}_i \in V_1$ or $x_i, \bar{x}_i \in V_2$.

If both $x_i$ and $\bar{x}_i$ are in $V_1$ then one can argue that there is no matching that saturates both $x_i$ and $\bar{x}_i$. If both $x_i$ and $\bar{x}_i$ are in $V_2$ then one can show that we end up deleting more than $k$ edges.

Now it is easy to see that for each vertex $y_i$, all copies $y_{i,1}, \ldots, y_{i,k+1}$ of it must be placed in the same partition as $y_i$ itself and hence all edges in $E_1, E_2, E_3$ lie across the cut $(V_1, V_2)$. Therefore the edges that were deleted from $G_\Phi$ were from $E_4$. Each of these edges corresponds to a distinct clause in $\Phi$. If a vertex $y_i$ is in $V_1$ assign the corresponding literal the value 1; else assign the literal the value 0. Note that this assignment is consistent as all copies of a vertex are in the same partition as the vertex itself and for no variable do we have that $x_i, \bar{x}_i \in V_1$ or $x_i, \bar{x}_i \in V_2$. This assignment satisfies all but the $k$ clauses that correspond to the edges that were deleted. ■

The parameterized complexity of both MEIKS and KEDS is open.

### 4.2 Approximation Results

For the MEIKS problem, it is easy to obtain a 2-approximation algorithm by simply finding a cut of size $m/2$ and then deleting all the other edges. In this subsection, we give a 4/3-approximation algorithm for graphs with a perfect matching and a 5/3-approximation algorithm for general graphs based on following combinatorial lemmas.

**Theorem 9.** *Let $G = (V, E)$ be a graph on $n$ vertices and $m$ edges with a perfect matching. Then $G$ has a subgraph with at least $\frac{3m}{4} + \frac{n}{8}$ edges that is König. This subgraph can, in fact, be found in time $O(n + m)$.*

*Proof.* Consider a perfect matching $P = \{e_1, \ldots, e_r\}$ of $G$, where $r = 2n$. We describe a randomized algorithm that constructs an edge-induced König subgraph with expected size $\frac{3m}{4} + \frac{n}{8}$. A König graph can be viewed as one in which the vertex set can be partitioned into two sets $V_1 \uplus V_2$, where $V_2$ is independent and there exists a matching across this partition which saturates $V_1$. Our randomized algorithm is as follows. From each edge $e_i = \{u_i, v_i\}$, select an endpoint of $e_i$ with probability 1/2 and place it in $V_1$ and keep the other endpoint in $V_2$. The probability that an edge is in $G[V_2]$ is 1/4. Also note that the probability that an edge of $P$ is in $G[V_2]$ is 0. For each edge $e$ of $G$, define $X_e$ to be a indicator random variable that takes the value 1, if $e$ is in $G[V_2]$ and 0 otherwise. Define $X = \sum_{i=1}^{m} X_i$. Then $E[X] = \sum_{i=1}^{m} E[X_i] = \frac{m-P}{4}$. Thus there exists a partition of the vertex set into $V_1 \uplus V_2$ such that the number of edges in $G[V_2]$ is at most $\frac{m-P}{4}$. Deleting these edges gives us a König graph with at least $\frac{3m}{4} + \frac{n}{8}$ edges. This algorithm can be easily derandomized by the method of conditional probabilities. This completes the proof. ■

For general graphs, we have the following result.

**Theorem 10.** *Let $G = (V, E)$ be a graph with a maximum matching $M$ and let $G_M = (V_M, E_M)$ be the graph induced on the vertices $V(M)$ of $M$. Then $G$ has an edge-induced König subgraph of size at least $\frac{3|E_M|}{4} + \frac{|E-E_M|}{2} + \frac{|M|}{4}$.*

*Proof.* Randomly partition the vertex set of $G$ into $V_1 \uplus V_2$ as follows. For each edge $e_i \in M$, select an endpoint of $e_i$ with probability 1/2 and place it in $V_1$. Define $V_2 = V - V_1$. Note that the edges in $M$ always lie across the cut $(V_1, V_2)$. An edge of $E_M - M$ is in $G[V_2]$ with probability 1/4; an edge in $E - E_M$ lies in $G[V_2]$ with probability 1/2. For each edge $e \in E$, define $X_e$ to be the indicator random variable that takes the value 1 if $e \in G[V_2]$ and 0 otherwise. Also define $X = \sum_{i=1}^{m} X_i$. Then $E[X] = \sum_{i=1}^{m} E[X_i] = \frac{|E_M - M|}{4} + \frac{|E-E_M|}{2}$. The claim follows. ■

**Theorem 11.** *Let $G = (V, E)$ be an undirected graph on $n$ vertices and $m$ edges. Then $G$ has an edge-induced König subgraph of size at least $\frac{3m}{5}$.*

*Proof.* Let $M$ be a maximum matching of $G$ and let $G[V_M] = (V_M, E_M)$ be the subgraph induced by the vertices $V(M)$ of $M$. Let $\eta(G)$ denote the size of the maximum edge induced König subgraph of $G$. By Theorem 10,

$$\eta(G) \geq \frac{3|E_M|}{4} + \frac{|E - E_M|}{2} + \frac{|M|}{4}. \quad (1)$$

Observe that by deleting all the edges in $G[V_M]$, we obtain a König subgraph of $G$. In fact, this is a bipartite graph with bipartition $V_M$ and $V - V_M$. Therefore,

$$\eta(G) \geq |E - E_M|. \tag{2}$$

Let $|E - E_M| = \alpha m$, where $\alpha > 0$ is a constant. Then by Equations 1 and 2, we have

$$\eta(G) \geq \max\left\{\alpha m, \frac{3(1-\alpha)m}{4} + \frac{\alpha m}{2}\right\}.$$

For a fixed $m$, the first function is monotonically increasing in $\alpha$ whereas the second is monotonically decreasing in $\alpha$. Equality holds when $\alpha m = \frac{3(1-\alpha)m}{4} + \frac{\alpha m}{2}$. A simple calculation shows that this happens when $\alpha = \frac{3}{5}$. This proves the claim. ■

The following theorem follows from Theorems 9 and 11 and the fact that the optimum König subgraph has at most $m$ edges.

**Theorem 12.** *For the optimization version of MEIKS problem, there exists a* $4/3$*-approximation algorithm for graphs with a perfect matching and a* $5/3$*-approximation algorithm for general graphs.*

### 4.3 FPT Algorithms

We give an FPT algorithm for the MEIKS problem by an application of exact algorithm for the optimization version of the problem and bounding the number of vertices and edges of the input graph as a linear function of $k$.

It is trivial to obtain an $O^*(2^m)$ algorithm for the optimization version of MEIKS problem. We next describe an $O^*(2^n)$ algorithm for this problem. To do this, we need a simple structural result characterizing minimal König edge deletion set of a graph.

**Theorem 13.** *Let $G = (V, E)$ be a graph. If $E'$ is a minimal König edge deletion set of $G$ then there exists $V' \subseteq V$ such that $E(G[V']) = E'$, that is, the edge set of the subgraph induced by $V'$ is precisely $E'$.*

*Proof.* Let $E'$ be a minimal König edge deletion set of $G$. Then $G' = (V, E - E')$ is König. Then the vertex set of $G'$ can be partitioned into $V_1$ and $V_2$ such that $V_2$ is a maximal independent set and there exists a matching saturating $V_1$ that lies across the cut $(V_1, V_2)$. We claim that $V' = V_2$. Since $E'$ is minimal, it is clear that $E(G[V_2]) = E'$. This completes the proof. ■

Our exact algorithm for the optimization version of MEIKS simply enumerates all possible subsets of the vertex set $V$ and for each subset $V' \subseteq V$, deletes all edges in $G[V']$ and checks whether the remaining graph is König. The algorithm returns an edge set $E' = E(G[V'])$ of smallest size such that $G - E'$ is König.

**Theorem 14.** *Given an $n$-vertex graph $G = (V, E)$, the optimization version of the* König Edge Deletion Set (KEDS) *(and hence the optimization version of MEIKS) can be solved in time $O^*(2^n)$ and space polynomial in $n$.*

**Theorem 15.** *The MEIKS problem can be solved in $O^*(2^k)$ time in connected undirected graphs.*

*Proof.* Let $(G, k)$ be an instance of the MEIKS problem, where $G$ is a graph with $m$ edges and $n$ vertices. If $k \leq 3m/5$ then answer yes and use the algorithm described in the previous subsection to actually obtain such a König subgraph. Otherwise, $m < 5k/3$. Also notice that any connected graph has a spanning tree, which is a bipartite graph and hence König graphs. Since number of edges in any tree is $n - 1$, if $k \leq n - 1$ then answer yes otherwise $n \leq k + 1$ and now we can apply Theorem 14 to obtain an $O^*(2^k)$ time algorithm for MEIKS problem. ■

## 5 Conclusion and Open Problems

In this paper, we introduced and studied vertex and edge versions of the König Subgraph problem from the points of view of parameterized complexity and approximation algorithms. There are several open problems including designing better approximation algorithms for the problems that we have considered in this paper. The other important open problems are whether the KVDS ($g$-VC) and KEDS problems are fixed parameter tractable.

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