# König Deletion Sets and Vertex Covers Above the Matching Size


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**Abstract.** A graph is König-Egerváry if the size of a minimum vertex cover equals the size of a maximum matching in the graph. We show that the problem of deleting at most $k$ vertices to make a given graph König-Egerváry is fixed-parameter tractable with respect to $k$. This is proved using interesting structural theorems on matchings and vertex covers which could be useful in other contexts.

We also show an interesting parameter-preserving reduction from the vertex-deletion version of red/blue-split graphs [4, 9] to a version of Vertex Cover and as a by-product obtain

1. the best-known exact algorithm for the optimization version of Odd Cycle Transversal [15];
2. fixed-parameter algorithms for several vertex-deletion problems including the following: deleting $k$ vertices to make a given graph (a) bipartite [17], (b) split [5], and (c) red/blue-split [7].


## 1 Introduction

The classical notions of *matchings* and *vertex covers* have been at the center of serious study for several decades in the area of Combinatorial Optimization [11]. In 1931, König and Egerváry independently proved a result of fundamental importance: in a bipartite graph the size of a maximum matching equals that of a minimum vertex cover [11]. This led to a polynomial-time algorithm for finding a minimum vertex cover in bipartite graphs. In fact, a maximum matching can be used to obtain a 2-approximation algorithm for the Minimum Vertex Cover problem in general graphs, which is still the best-known approximation algorithm for this problem. Interestingly, this min-max relationship holds for a larger class of graphs known as König-Egerváry graphs and it includes bipartite graphs as a proper subclass. König-Egerváry graphs will henceforth be called König graphs.

König graphs have been studied for a fairly long time from a structural point of view [1, 3, 9, 10, 18, 15]. Both Deming [3] and Sterboul [18] gave independent characterizations of König graphs and showed that König graphs can be recognized in polynomial time. Lovász [10] used the theory of matching-covered graphs to give an excluded-subgraph characterization of König graphs that contain a perfect matching. Korach et al. [9] generalized this and gave an excluded-subgraph characterization for the class of all König graphs.

A natural optimization problem associated with a graph class $\mathcal{G}$ is the following: given a graph $G$, what is the minimum number of vertices to be deleted from $G$ to obtain a graph in $\mathcal{G}$? For example, when $\mathcal{G}$ is the class of empty graphs, forests or bipartite graphs, the corresponding problems are Vertex Cover, Feedback Vertex Set and Odd Cycle Transversal, respectively. We call the vertex-deletion problem corresponding to class of König graphs the König Vertex Deletion problem. A set of vertices whose deletion makes a given graph König is called a König vertex deletion set. In the parameterized setting, the parameter for vertex-deletion problems is the solution size, that is, the number of vertices to be deleted so that the resulting graph belongs to the given graph class.

An algorithmic study of the König Vertex Deletion problem was initiated in [12], where it was shown that when restricted to the class of graphs with a perfect matching, König Vertex Deletion fixed-parameter reduces to a problem known as Min 2-Cnf Deletion. This latter problem was shown to be fixed-parameter tractable by Razgon and O'Sullivan [16]. This immediately implies that König Vertex Deletion is fixed-parameter tractable for graphs with a perfect matching. But the parameterized complexity of the problem in general graphs remained open.

In this paper, we first establish interesting structural connections between minimum vertex covers and minimum König vertex deletion sets. Using these, we show that

1. the parameterized König Vertex Deletion problem is fixed-parameter tractable, and
2. there exists an $O^*(1.221^n)$ algorithm[4] for the optimization version of König Vertex Deletion problem, where $n$ denotes the number of vertices in the graph.

Note that König graphs are not hereditary, that is, not closed under taking induced subgraphs. For instance, a 3-cycle is not König but attaching an edge to one of the vertices of the 3-cycle results in a König graph. In fact, König Vertex Deletion is one of the few vertex-deletion problems associated with a non-hereditary graph class whose parameterized complexity has been resolved. Another such example can be found in [13].

Our second result is an interesting parameter-preserving reduction from the vertex-deletion version of red/blue-split graphs [4, 9] to a version of Vertex Cover called Above Guarantee Vertex Cover. A red/blue graph [7] is a tuple $(G = (V, E), c)$, where $G = (V, E)$ is a simple undirected graph and $c : E \rightarrow 2^{\{r,b\}} \setminus \emptyset$ is an assignment of "colors" red and blue to the edges of the graph. An edge may be assigned both red and blue simultaneously and we require that $R$, the set of red edges, and $B$, the set of blue edges, both be nonempty. A red/blue graph $G = (V, R \cup B)$ is red/blue-split if its vertex set can be partitioned into a red independent set $V_R$ and a blue independent set $V_B$. A red (resp. blue) independent set is an independent set in the red graph $G_R = (V, R)$ (resp. blue graph $G_B = (V, B)$). A graph $G$ is split if its vertex set can be partitioned into an

[4] The $O^*(\cdot)$ notation suppresses polynomial terms. We write $O^*(T(n))$ for a time complexity of the form $O(T(n) \cdot \text{poly}(n))$, where $T(n)$ grows exponentially with $n$.

independent set and a clique. A 2-clique graph is a graph whose vertex set can be partitioned into two cliques. Note that a graph is 2-clique if and only if it is the complement of a bipartite graph. We will see that red/blue-split graphs are a generalization of König (and hence bipartite) graphs, split and 2-clique graphs [7].

As a by-product of the reduction from Red/Blue-Split Vertex Deletion to Above Guarantee Vertex Cover we obtain:

1. an $O^*(1.49^n)$ algorithm for the optimization version of Red/Blue-Split Vertex Deletion, Odd Cycle Transversal, Split Vertex Deletion and 2-Clique Vertex Deletion.[5]
2. fixed parameter algorithms for all the above problems.

For Odd Cycle Transversal, this gives the best-known exact algorithm for the optimization version improving over the previous best of $O^*(1.62^n)$ [15].

This paper is organized as follows. In Section 2 we give a brief outline of parameterized complexity, the notations and known results that we use in the rest of the paper. In Section 3 we show the König Vertex Deletion problem to be fixed-parameter tractable. In Section 4 we show that a number of vertex-deletion problems fixed-parameter reduce to Red/Blue-Split Vertex Deletion which fixed-parameter reduces to Above Guarantee Vertex Cover. Finally in Section 5 we end with some concluding remarks and directions for further research.

## 2 Preliminaries

In this section we summarize the necessary concepts concerning parameterized complexity, fix our notation and outline some results that we make use of in the paper.

### 2.1 Parameterized Complexity

A parameterized problem is a subset of $\Sigma^* \times \mathbb{Z}^{\geq 0}$, where $\Sigma$ is a finite alphabet and $\mathbb{Z}^{\geq 0}$ is the set of nonnegative numbers. An instance of a parameterized problem is therefore a pair $(I, k)$, where $k$ is the parameter. In the framework of parameterized complexity, the running time of an algorithm is viewed as a function of two quantities: the size of the problem instance *and* the parameter. A parameterized problem is said to be *fixed-parameter tractable (FPT)* if there exists an algorithm that takes as input $(I, k)$ and decides whether it is a yes or no-instance in time $O(f(k) \cdot |I|^{O(1)})$, where $f$ is a function depending only on $k$. The class FPT consists of all fixed parameter tractable problems.

A parameterized problem $\pi_1$ is *fixed-parameter reducible* to a parameterized problem $\pi_2$ if there exist functions $f, g : \mathbb{Z}^{\geq 0} \to \mathbb{Z}^{\geq 0}$, $\Phi : \Sigma^* \times \mathbb{Z}^{\geq 0} \to \Sigma^*$ and a polynomial $p(\cdot)$ such that for any instance $(I, k)$ of $\pi_1$, $(\Phi(I, k), g(k))$ is an instance of $\pi_2$ computable in time $f(k) \cdot p(|I|)$ and $(I, k) \in \pi_1$ if and only if $(\Phi(I, k), g(k)) \in \pi_2$. Two parameterized problems are *fixed-parameter equivalent* if they are fixed-parameter reducible to each other. The basic complexity class for fixed-parameter intractability is $W[1]$ as

[5] Since a 2-clique graph is the complement of a bipartite graph, the 2-Clique Vertex Deletion problem is NP-complete [17].

there is strong evidence that $W[1]$-hard problems are not fixed-parameter tractable. To show that a problem is $W[1]$-hard, one needs to exhibit a fixed-parameter reduction from a known $W[1]$-hard problem to the problem at hand. For more on parameterized complexity see [14].

### 2.2 Notation

Given a graph $G$, we use $\mu(G)$, $\beta(G)$ and $\kappa(G)$ to denote, respectively, the size of a maximum matching, a minimum vertex cover and a minimum König vertex deletion set of $G$. When the graph being referred to is clear from the context, we simply use $\mu$, $\beta$ and $\kappa$. Given a graph $G = (V, E)$ and two disjoint vertex subsets $V_1, V_2$ of $V$, we let $(V_1, V_2)$ denote the bipartite graph with vertex set $V_1 \cup V_2$ and edge set $\{\{u, v\} : \{u, v\} \in E \text{ and } u \in V_1, v \in V_2\}$. If $B$ is a bipartite graph with vertex partition $L \uplus R$ then we let $\mu(L, R)$ denote the size of the maximum matching of $B$. If $M$ is matching and $\{u, v\} \in M$ then we say that *$u$ is the partner of $v$ in $M$*. If the matching being referred to is clear from the context we simply say *$u$ is a partner of $v$*. The vertices of $G$ that are the endpoints of edges in the matching $M$ are said to be *saturated by $M$*; all other vertices are *unsaturated by $M$*.

### 2.3 Related Results

We next mention some known results about König graphs and the Above Guarantee Vertex Cover problem.

**Fact 1** *(See for instance* [12].*) A graph $G = (V, E)$ is König if and only if there exists a polynomial-time algorithm that partitions $V(G)$ into $V_1$ and $V_2$ such that $V_1$ is a minimum vertex cover of $G$ and there exists a matching across the cut $(V_1, V_2)$ saturating every vertex of $V_1$.*

Given a graph $G$ it is clear that $\beta(G) \geq \mu(G)$. The Above Guarantee Vertex Cover problem is this: given a graph $G$ and an integer parameter $k$ decide whether $\beta(G) \leq \mu(G)+k$. As was shown in [12], for this problem we may assume that the input graph $G = (V, E)$ has a perfect matching.

**Theorem 1.** [12] *Let $G = (V, E)$ be a graph with a maximum matching $M$ and let $I := V \setminus V(M)$. Construct $G'$ by replacing every vertex $u \in I$ by a vertex pair $u, u'$ and adding the edges $\{u, u'\}$ and $\{u', v\}$ for all $\{u, v\} \in E$. Then $G$ has a vertex cover of size $\mu(G) + k$ if and only if $G'$ has a vertex cover of size $\mu(G') + k$.*

In [12], we also showed that the Above Guarantee Vertex Cover problem fixed-parameter reduces to Min 2-Sat Del which is the problem of deciding whether $k$ clauses can be deleted from a given 2-Cnf Sat formula to make it satisfiable. Since Min 2-Sat Del is fixed-parameter tractable [16], so is Above Guarantee Vertex Cover. The algorithm for Above Guarantee Vertex Cover actually outputs a vertex cover of size $\mu(G)+k$ if there exists one.

**Corollary 1.** *Given a graph $G = (V, E)$ and an integer $k$, one can decide whether $G$ has a vertex cover of size at most $\mu(G) + k$ in time $O(15^k \cdot k \cdot |E|^3)$. If $G$ has a vertex cover of size $\mu(G) + k$ then the algorithm actually outputs one such vertex cover.*

Note that Theorem 1 says that for the Above Guarantee Vertex Cover problem it is sufficient to consider graphs with a *perfect matching*. In [12], we showed that the König Vertex Deletion problem on graphs with a perfect matching fixed-parameter reduces to Above Guarantee Vertex Cover. This shows that König Vertex Deletion on graphs with a perfect matching is fixed-parameter tractable. However this does not seem to resolve the parameterized complexity status of König Vertex Deletion in general graphs. We do not know of a fixed-parameter reduction from the general case to the case with a perfect matching as in the case of Above Guarantee Vertex Cover. However, in the next section, we show the general problem to be fixed-parameter tractable using some new structural results between maximum matchings and vertex covers.

## 3 The König Vertex Deletion Problem

We now consider the König Vertex Deletion Problem in general graphs and show it fixed-parameter tractable.

Suppose $Y$ is a vertex cover in a graph $G = (V, E)$. Consider a maximum matching $M$ between $Y$ and $V \setminus Y$. If $M$ saturates every vertex of $Y$ then the graph is König. If not, then $Y \setminus V(M)$, the set of vertices of $Y$ unsaturated by $M$, is a König deletion set by Fact 1. What we prove in this section is that if $Y$ is a minimum vertex cover, then $Y \setminus V(M)$ is a minimum König vertex deletion set.

Our first observation is that any minimum König vertex deletion set is contained in some minimum vertex cover.

**Theorem 2.** *Let $G$ be an undirected graph with a minimum König vertex deletion set $K$. Let $V(G \setminus K) = V_1 \uplus V_2$ where $V_2$ is independent and there is a matching $M$ from $V_1$ to $V_2$ saturating $V_1$. Then $V_1 \cup K$ is a minimum vertex cover for $G$.*

*Proof.* Suppose $S$ is a vertex cover of $G$ such that $|S| < |V_1| + |K|$. We will show that there exists a König vertex deletion set of size smaller than $|K|$, contradicting our hypothesis. Define $V_1' = V_1 \cap S$, $V_2' = V_2 \cap S$ and $K' = K \cap S$. Let $A_1$ be the vertices of $V_1'$ whose partner in $M$ is in $V_2'$ and let $A_2$ be the vertices of $V_1'$ whose partner in $M$ is not in $V_2'$. See Figure 1. We claim that $A_1 \cup K'$ is a König vertex deletion set of $G$ and

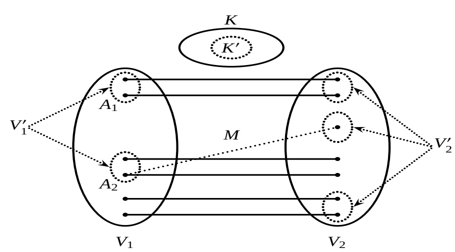


**Fig. 1.** The sets that appear in the proof of Theorem 2. The matching $M$ consists of the solid edges across $V_1$ and $V_2$.

$|A_1 \cup K'| < |K|$, which will produce the required contradiction and prove the theorem. This claim is proved using the following three claims:

*Claim 1.* $|A_1 \cup K'| < |K|$.

*Claim 2.* $A_2 \cup V_2'$ *is a vertex cover in* $G \setminus (A_1 \cup K')$.

*Claim 3. There exists a matching between* $A_2 \cup V_2'$ *and* $V \setminus (V_1' \cup K' \cup V_2')$ *saturating every vertex of* $A_2 \cup V_2'$.

*Proof of Claim 1.* Clearly $|S| = |V_1'| + |V_2'| + |K'|$. Note that $S$ intersects $|A_1|$ of the edges of $M$ in both end points and $|M| - |A_1|$ edges of $M$ in one end point (in either $V_1'$ or $V_2'$). Furthermore $V_2'$ has $|V_2' \setminus V(M)|$ vertices of $S$ that do not intersect any edge of $M$. Hence $|M| + |A_1| + |V_2' \setminus V(M)| = |V_1'| + |V_2'|$. That is, $|V_1'| + |V_2'| = |V_1| + |A_1| + |V_2' \setminus V(M)|$ (as $|M| = |V_1|$). Hence $|S| < |V_1| + |K|$ implies that $|A_1| + |V_2' \setminus V(M)| + |K'| < |K|$ which implies that $|A_1| + |K'| < |K|$ proving the claim.

*Proof of Claim 2.* Since $S = A_1 \cup A_2 \cup V_2' \cup K'$ is a vertex cover of $G$, clearly $A_2 \cup V_2'$ covers all edges in $G \setminus (A_1 \cup K')$.

*Proof of Claim 3.* Since the partner of a vertex in $A_2$ in $M$ is in $V \setminus (V_1' \cup K' \cup V_2')$, we can use the edges of $M$ to saturate vertices in $A_2$. To complete the proof, we show that in the bipartite graph $(V_2', (V_1 \setminus V_1') \cup (K \setminus K'))$ there is a matching saturating $V_2'$. To see this, note that any subset $D \subseteq V_2'$ has at least $|D|$ neighbors in $(V_1 \setminus V_1') \cup (K \setminus K')$. For otherwise, let $D'$ be the set of neighbors of $D$ in $(V_1 \setminus V_1') \cup (K \setminus K')$ where we assume $|D| > |D'|$. Then $(S \setminus D) \cup D'$ is a vertex cover of $G$ of size strictly less than $|S|$, contradicting the fact that $S$ is a minimum vertex cover. To see that $(S \setminus D) \cup D'$ is indeed a vertex cover of $G$, note that $S \setminus V_2'$ covers all edges of $G$ except those in the graph $(V_2', (V_1 \setminus V_1') \cup (K \setminus K'))$ and all these edges are covered by $(V_2' \setminus D) \cup D'$. Hence by Hall's theorem, there exists a matching saturating all vertices of $V_2'$ in the bipartite graph $(V_2', (V_1 \setminus V_1') \cup (K \setminus K'))$, proving the claim.

This completes the proof of the theorem. □

Theorem 2 has interesting consequences.

**Corollary 2.** *For any two minimum König vertex deletion sets (KVDSs)* $K_1$ *and* $K_2$, $\mu(G \setminus K_1) = \mu(G \setminus K_2)$.

*Proof.* Since $K_1$ is a minimum KVDS of $G$, $\beta(G \setminus K_1) = \mu(G \setminus K_1)$. By Theorem 2, $\beta(G \setminus K_1) + |K_1| = \beta(G)$ and $\beta(G \setminus K_2) + |K_2| = \beta(G)$. Since $|K_1| = |K_2|$, it follows that $\beta(G \setminus K_1) = \beta(G \setminus K_2)$ and hence $\mu(G \setminus K_1) = \mu(G \setminus K_2)$. □

From Theorem 2 and Fact 1, we get

**Corollary 3.** *Given a graph* $G = (V, E)$ *and a minimum König vertex deletion set for* $G$*, one can construct a minimum vertex cover for* $G$ *in polynomial time.*

Our goal now is to prove the "converse" of Corollary 3. In particular, we would like to construct a minimum König vertex deletion set from a minimum vertex cover. Our first step is to show that if we know that a given minimum vertex cover contains a minimum König vertex deletion set then we can find the König vertex deletion set in polynomial time. Recall that given a graph $G = (V, E)$ and $A, B \subseteq V$ such that $A \cap B = \emptyset$, we use $\mu(A, B)$ to denote a maximum matching in the bipartite graph comprising of the vertices in $A \cup B$ and the edges in $\{\{u, v\} \in E : u \in A, v \in B\}$. We denote this graph by $(A, B)$.

**Lemma 1.** *Let $K$ be a minimum KVDS and $Y$ a minimum vertex cover of a graph $G = (V, E)$ such that $K \subseteq Y$. Then $\mu(G \setminus K) = \mu(Y, V \setminus Y)$ and $|K| = |Y| - \mu(Y, V \setminus Y)$.*

*Proof.* If $G$ is König then the theorem clearly holds. Therefore assume that $K \neq \emptyset$. Note that $Y \setminus K$ is a minimum vertex cover of the König graph $G \setminus K$. Thus $\mu(G \setminus K) = \mu(Y \setminus K, V \setminus Y)$. We claim that $\mu(Y \setminus K, V \setminus Y) = \mu(Y, V \setminus Y)$. For if not, we must have $\mu(Y \setminus K, V \setminus Y) < \mu(Y, V \setminus Y)$. Then let $M$ be a maximum matching in the bipartite graph $(Y, V \setminus Y)$ and $K' \subseteq Y$ be the set of vertices unsaturated by $M$. Note that $K' \neq \emptyset$ is a KVDS for $G$. Since $\mu(Y, V \setminus Y) = |Y| - |K'|$ and $\mu(Y \setminus K, V \setminus Y) = |Y| - |K|$ we have $|K'| < |K|$, a contradiction, since by hypothesis $K$ is a smallest KVDS for $G$. Therefore we must have $\mu(G \setminus K) = \mu(Y, V \setminus Y)$ and $|K| = |Y| - \mu(Y, V \setminus Y)$. □

The next lemma says that $\mu(Y, V \setminus Y)$ is the same for all minimum vertex covers $Y$ of a graph $G$. Together with Lemma 1, this implies that if $K$ is a minimum König vertex deletion set and $Y$ is a minimum vertex cover of a graph $G = (V, E)$, then $\mu(G \setminus K) = \mu(Y, V \setminus Y)$. This result is crucial to our FPT-algorithm for the König Vertex Deletion problem.

**Lemma 2.** *For any two minimum vertex covers $Y_1$ and $Y_2$ of $G$, $\mu(Y_1, V \setminus Y_1) = \mu(Y_2, V \setminus Y_2)$.*

*Proof.* Suppose without loss of generality that $\mu(Y_1, V \setminus Y_1) > \mu(Y_2, V \setminus Y_2)$. Let $M_1$ be a maximum matching in the bipartite graph $(Y_1, V \setminus Y_1)$. To arrive at a contradiction, we study how $Y_2$ intersects the sets $Y_1$ and $V \setminus Y_1$ with respect to the matching $M_1$. To this end, we define the following sets (see Figure 2):

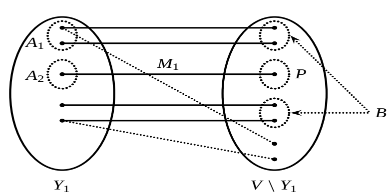


**Fig. 2.** The sets that appear in the proof of Lemma 2. The solid edges across $Y_1$ and $V \setminus Y_1$ constitute the matching $M_1$.

- $A = Y_2 \cap Y_1 \cap V(M_1)$.
- $B = Y_2 \cap (V \setminus Y_1) \cap V(M_1)$.
- $A_1$ is the set of vertices in $A$ whose partners in $M_1$ are also in $Y_2$.
- $A_2$ is the set of vertices in $A$ whose partners in $M_1$ are not in $Y_2$.

We first show that

*Claim.* In the bipartite graph $(Y_2, V \setminus Y_2)$ there is a matching saturating each vertex in $A_2 \cup B$.

It will follow from the claim that $\mu(Y_2, V \setminus Y_2) \geq |A_2| + |B|$. However, note that $Y_2$ intersects every edge of $M_1$ at least once (as $Y_2$ is a vertex cover). More specifically, $Y_2$ intersects $|A_1|$ edges of $M_1$ twice and $|M_1| - |A_1|$ edges once (either in $Y_1$ or in $V \setminus Y_1$). Hence, $|A|+|B| = |M_1|+|A_1|$ and so $|A_2|+|B| = |M_1|$ and so $\mu(Y_2, V \setminus Y_2) \geq |A_2|+|B| = |M_1|$ a contradiction to our assumption at the beginning of the proof. Thus it suffices to prove the claim.

*Proof of Claim.* Let $P$ denote the partners of the vertices of $A_2$ in $M_1$. Since $P \subseteq V \setminus Y_2$, we use the edges of $M_1$ to saturate vertices of $A_2$. Hence it is enough to show that the bipartite graph $\mathcal{B} = (B, (V \setminus Y_2) \setminus P)$ contains a matching saturating the vertices in $B$. Suppose not. By Hall's Theorem there exists a set $D \subseteq B$ such that $|N_{\mathcal{B}}(D)| < |D|$. We claim that the set $Y_2' := Y_2 \setminus D + N_{\mathcal{B}}(D)$ is a vertex cover of $G$. To see this, note that the vertices in $Y_2 \setminus D$ cover all the edges of $G$ except those in the bipartite graph $(D, Y_1 \cap (V \setminus Y_2))$ and these are covered by $N_{\mathcal{B}}(D)$. Therefore $Y_2'$ is a vertex cover of size strictly smaller than $Y_2$, a contradiction. This proves that there exists a matching in $(Y_2, V \setminus Y_2)$ saturating each vertex in $A_2 \cup B$.

This completes the proof of the lemma. □

The next theorem shows how we can obtain a minimum König vertex deletion set from a minimum vertex cover in polynomial time.

**Theorem 3.** *Given a graph $G = (V, E)$, let $Y$ be any minimum vertex cover of $G$ and $M$ a maximum matching in the bipartite graph $(Y, V \setminus Y)$. Then $K := Y \setminus V(M)$ is a minimum König vertex deletion set of $G$.*

*Proof.* Clearly $K$ is a KVDS. Let $K_1$ be a minimum KVDS of $G$. By Theorem 2, there exists a minimum vertex cover $Y_1$ such that $K_1 \subseteq Y_1$ and

$$\begin{aligned}
|K_1| &= |Y_1| - \mu(Y_1, V \setminus Y_1) && \text{(By Lemma 1.)} \\
&= |Y| - \mu(Y_1, V \setminus Y_1) && \text{(Since } Y_1 \text{ and } Y \text{ are minimum vertex covers.)} \\
&= |Y| - \mu(Y, V \setminus Y) && \text{(By Lemma 2.)} \\
&= |K|
\end{aligned}$$

This proves that $K$ is a minimum KVDS. □

**Corollary 4.** *Given a graph $G = (V, E)$ and a minimum vertex cover for $G$, one can construct a minimum König vertex deletion set for $G$ in polynomial time.*

Note that although both these problems–Vertex Cover and König Vertex Deletion Set–are NP-complete, we know of very few pairs of such parameters where we can obtain one from the other in polynomial time (e.g. edge dominating set and minimum maximal matching, see [8]). In fact, there are parameter-pairs such as dominating set and vertex cover where such a polynomial-time transformation is not possible unless P = NP. This follows since in bipartite graphs, for instance, a minimum vertex cover is computable in polynomial time whereas computing a minimum dominating set is NP-complete.

To show that the König Vertex Deletion problem is fixed-parameter tractable we make use of the following

**Lemma 3.** [12] *If $G$ is a graph such that $\beta(G) = \mu(G) + k$, then $k \leq \kappa(G) \leq 2k$.*

We are now ready to prove that the König Vertex Deletion problem is fixed-parameter tractable in general graphs.

**Theorem 4.** *Given a graph $G = (V, E)$ and an integer parameter $k$, the problem of whether $G$ has a subset of at most $k$ vertices whose deletion makes the resulting graph König can be decided in time $O(15^k \cdot k^2 \cdot |E|^3)$.*

*Proof.* Use the FPT algorithm from Corollary 1 to test whether $G$ has a vertex cover of size at most $\mu(G) + k$. If not, by Lemma 3, we know that the size of a minimum König vertex deletion set is strictly more than $k$. Therefore return no. If yes, then find the size of a minimum vertex cover by applying Corollary 1 with every integer value between 0 and $k$ for the excess above $\mu(G)$. Note that for yes-instances of the Above Guarantee Vertex Cover problem, the FPT algorithm actually outputs a vertex cover of size $\mu(G) + k$. We therefore obtain a minimum vertex cover of $G$. Use Theorem 3 to get a minimum König vertex deletion set in polynomial time and depending on its size answer the question. It is easy to see that all this can be done in time $O(15^k \cdot k^2 \cdot |E|^3)$. □

We know that computing a maximum independent set (or equivalently a minimum vertex cover) in an $n$-vertex graph can be done in time $O^*(2^{0.288n})$ [6]. By Corollary 4, we can compute a minimum König vertex deletion set in the same exponential time. Given a graph $G$ together with a tree-decomposition for it of width $w$, one can obtain a minimum vertex cover in time $O^*(2^w)$ [14]. For the definitions of treewidth and tree-decomposition, refer [14]. In general, algorithms on graphs of bounded treewidth are based on dynamic programming over the tree-decomposition of the graph. It is not obvious how to find such a dynamic programming algorithm for the König Vertex Deletion problem. By applying Corollary 4, we can find a minimum König deletion set in time $O^*(2^w)$ in graphs of treewidth $w$. The above discussion results in the following corollary.

**Corollary 5.**
1. *Given a graph $G = (V, E)$ on $n$ vertices we can find a minimum König vertex deletion set in time $O^*(2^{0.288n}) = O^*(1.221^n)$.*
2. *If a tree-decomposition for $G$ of width $w$ is given, we can find a minimum König vertex deletion set in time $O^*(2^w)$.*

## 4 Red/Blue-Split Graphs and Above Guarantee Vertex Cover

In this section we introduce the Red/Blue-Split Vertex Deletion problem and show that a number of vertex-deletion problems fixed-parameter reduce to it. Recall that a red/blue-graph is one in which the edges are colored red or blue and where an edge may receive multiple colors. A red/blue-graph $G = (V, R \cup B)$ is red/blue-split if its vertex set can be partitioned into a red independent set and a blue independent set, where a red (blue) independent set is an independent set in the red graph $G = (V, R)$ (blue graph $G = (V, B)$). In what follows we use r/b as an abbreviation for red/blue and $E_c$ to denote the set of edges assigned color $c$.

The R/B-Split Vertex Deletion problem is the following: given an r/b-graph $G = (V, R \cup B)$ and an integer $k$, are there $k$ vertices whose deletion makes $G$ r/b-split? We first show that R/B-Split Vertex Deletion fixed-parameter reduces to the Above Guarantee Vertex Cover problem. Since Above Guarantee Vertex Cover is fixed-parameter tractable, this will show that R/B-Split Vertex Deletion is fixed-parameter tractable too.

At this point, we note that r/b-split graphs can be viewed as a generalization of König graphs as follows. A graph $G = (V, E)$ with a maximum matching $M$ is König if and only if the 2-colored graph $G' = (V, R \cup B)$, where $R = E$ and $B = M$, is r/b-split. It is important to realize that while this gives a recognition algorithm for König graphs using one for r/b-split graphs, it does not seem to give any relationship between the corresponding vertex-deletion problems. In fact, we do not know of any parameter-preserving reduction from König Vertex Deletion to the R/B-Split Vertex Deletion problem for general graphs.

For graphs with a perfect matching, we show by an independent argument based on the structure of a minimum König vertex deletion set that the König Vertex Deletion problem does indeed fixed-parameter reduce to Above Guarantee Vertex Cover (and also to R/B-Split Vertex Deletion) [12]. But this structural characterization of minimum König vertex deletion sets does not hold in general graphs. Therefore the fixed-parameter tractability result for König Vertex Deletion Set cannot be obtained from that of R/B-Split Vertex Deletion.

**Theorem 5.** *Let $G = (V, E = E_r \cup E_b)$ be an r/b graph. Construct $G' = (V', E')$ as follows: the vertex set $V'$ consists of two copies $V_1, V_2$ of $V$ and for all $u \in V$, $u_1 \in V_1$ and $u_2 \in V_2$ the edge set $E' = \{\{u_1, u_2\} : u \in V\} \cup \{\{u_1, v_1\} : \{u, v\} \in E_r\} \cup \{\{u_2, v_2\} : \{u, v\} \in E_b\}$. Then there exists $k$ vertices whose deletion makes $G$ r/b-split if and only if $G'$ has a vertex cover of size $\mu(G') + k$.*

*Proof.* Clearly $G'$ has $2|V|$ vertices and a perfect matching of size $|V|$. It suffices to show that $G$ has an r/b-split subgraph on $t$ vertices if and only if $G'$ has an independent set of size $t$. This would prove that there exists $|V| - t$ vertices whose deletion makes $G$ r/b-split if and only if $G'$ has a vertex cover of size $2|V| - t$. Finally, plugging in $k = |V| - t$ will prove the theorem.

Therefore let $H$ be an r/b-split subgraph of $G$ on $t$ vertices with a red independent set $V_r$ and a blue independent set $V_b$. Then the copy $V_r^1$ of $V_r$ in $V_1$ and the copy $V_b^2$ of $V_b$ in $V_2$ are independent sets in $G'$. Since $V_r \cap V_b = \emptyset$, $V_r^1 \cup V_b^2$ is an independent set in $G'$ on $t$ vertices. Conversely if $H'$ is an independent set in $G'$ of size $t$, then for $i = 1, 2$ let $V(H') \cap V_i = W_i$ and $|W_i| = t_i$ so that $t_1 + t_2 = t$. For $i = 1, 2$, let $\widetilde{W_i}$ be the vertices in $V(G)$ corresponding to the vertices in $W_i$. Then $\widetilde{W_1}$ is an independent set of size $t_1$ in the red graph $G_r = (V(G), E_r)$ and $\widetilde{W_2}$ is an independent set of size $t_2$ in the blue graph $G_b = (V(G), E_b)$. Since $W_1$ and $W_2$ do not both contain copies of the same vertex of $V(G)$, as $W_1 \cup W_2$ is independent, we have $\widetilde{W_1} \cap \widetilde{W_2} = \emptyset$. Thus $G[\widetilde{W_1} \cup \widetilde{W_2}]$ is an r/b-split graph of size $t$ in $G$. □

Since a maximum independent set in an $n$-vertex graph can be obtained in time $O^*(2^{0.288n})$ [6], we immediately have the following

**Corollary 6.** *The optimization version of the* R/B-Split Vertex Deletion *problem can be solved in time $O^*(2^{0.576n}) = O^*(1.49^n)$ on input graphs on n vertices.*

Since Above Guarantee Vertex Cover is fixed-parameter tractable (Corollary 1) we have

**Corollary 7.** *The parameterized version of the* R/B-Split Vertex Deletion *problem is fixed-parameter tractable and can be solved in time $O(15^k \cdot k^2 \cdot m^3)$, where m is the number of edges in the input graph.*

As mentioned before, König (and hence bipartite) and split graphs can be viewed as r/b-split graphs. Since 2-clique graphs are complements of bipartite graphs it follows that they can also be viewed as r/b-split graphs. The vertex-deletion problems Odd Cycle Transversal, Split Vertex Deletion and 2-Clique Vertex Deletion fixed-parameter reduce to R/B-Split Vertex Deletion. We show this reduction for Odd Cycle Transversal as the proofs in the other cases are quite similar.

**Theorem 6.** *Given a simple undirected graph $G = (V, E)$, construct an r/b-graph $G' = (V', E')$ as follows: define $V' = V$ and $E' = E$; $E_r(G') = E$ and $E_b(G') = E$. Then there exists k vertices whose deletion makes G bipartite if and only if there exist k vertices whose deletion makes $G'$ r/b-split.*

*Proof.* Suppose deleting $k$ vertices from $G$ makes it bipartite with vertex bipartition $V_1 \cup V_2$. Then $V_1$ and $V_2$ are independent in both the red graph $G'_r = (V', E_r)$ and in the blue graph $G'_b = (V', E_b)$. Thus $G'[V_1 \cup V_2]$ is r/b-split. Conversely if $k$ vertices can be deleted from $G'$ to make it r/b-split, let $V_r$ and $V_b$ be the red and blue independent sets respectively. Then both these sets must be independent in $G$ and therefore the subgraph of $G$ induced on $V_r \cup V_b$ is bipartite. □

From Theorem 6 and Corollaries 6 and 7 the following result follows immediately.

**Corollary 8.** *The parameterized version of* Odd Cycle Transversal, Split Vertex Deletion *and* 2-Clique Vertex Deletion *are fixed-parameter tractable and their optimization versions can be solved in time $O^*(2^{0.576n}) = O^*(1.49^n)$ on input graphs on n vertices.*

## 5 Conclusion

We showed that the König Vertex Deletion problem is fixed-parameter tractable in general graphs. To prove this, we made use of a number of structural results involving minimum vertex covers, minimum König vertex deletion sets and maximum matchings. We also showed that a number of vertex-deletion problems, in particular, R/B-Split Vertex Deletion, and Odd Cycle Transversal fixed-parameter reduce to Above Guarantee Vertex Cover. Since the latter problem is FPT, all these vertex-deletion problems are also FPT.

An interesting open problem is the parameterized complexity of the König Edge Deletion problem: given $G = (V, E)$ and an integer parameter $k$, does there exist at most $k$ edges whose deletion makes the resulting graph König? Deriving a problem kernel for König Vertex Deletion Set is an interesting open problem. Another natural open problem to design better FPT algorithms for Above Guarantee Vertex Cover perhaps without using the reduction to Min 2-Sat Deletion.

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## References

1. J .M. Bourjolly and W. R. Pulleyblank. *König-Egerváry Graphs, 2-Bicritical Graphs and Fractional Matchings.* Disc. Appl. Math. Vol. 24, pp. 63-82, 1989.
2. L. Cai. *Fixed-Parameter Tractability of Graph Modification Problems for Hereditary Properties.* Inform. Proc. Lett. Vol. 58, Issue 4, pp. 171-176, 1996.
3. R. W. Deming. *Independence Numbers of Graphs – An Extension of the König-Egerváry Theorem.* Disc. Math. Vol. 27, pp. 23-33, 1979.
4. U. Faigle, B. Fuchs and B. Wienand. *Covering Graphs by Colored Stable Sets.* Electronic Notes in Disc. Math. Vol. 17, pp. 145-149, 2004.
5. S. Földes and P. L. Hammer. *Split graphs.* In Proc. of the 8th Southeastern Conference on Combinatorics, Graph Theory and Computing (Louisiana State Univ., Baton Rouge, La., 1977), pp. 311-315, Winnipeg: Congressus Numerantium, No. XIX, Utilitas Math.
6. F. V. Fomin, F. Grandoni and D. Kratsch. *Measure and Conquer: A Simple $O(2^{0.288n})$ Independent Set Algorithm.* In Proc. of SODA 2006, pp. 18-25, 2006.
7. F. Gavril. *An Efficiently Solvable Graph Partition Problem to Which Many Problems are Reducible.* Inform. Proc. Lett. Vol. 45, pp. 285-290, 1993.
8. F. Gavril and M. Yannakakis. *Edge Dominating Sets in Graphs.* SIAM Jour. Appl. Math. Vol. 38, Issue 3, pp. 364-372, 1980.
9. E. Korach, T. Nguyen and B. Peis. *Subgraph Characterization of Red/Blue-Split Graphs and König-Egerváry Graphs.* In the Proc. of SODA 2006, pp. 842-850, 2006.
10. L. Lovász. *Ear-Decompositions of Matching-covered Graphs.* Combinatorica Vol. 3, pp. 105-118, 1983.
11. L. Lovász and M. D. Plummer. *Matching Theory.* North Holland, 1986.
12. S. Mishra, V. Raman, S. Saurabh, S. Sikdar and C. R. Subramanian. *The Complexity of Finding Subgraphs Whose Matching Number Equals the Vertex Cover Number.* In Proc. of ISAAC 2007, Springer LNCS Vol. 4835, pp. 268-279, 2007.
13. H. Moser and D. M. Thilikos. *Parameterized Complexity of Finding Regular Induced Subgraphs.* In Proc. of ACiD 2006, pp. 107-118.
14. R. Niedermeier. *An Invitation to Fixed-Parameter Algorithms.* Oxford University Press, 2006.
15. V. Raman, S. Saurabh and S. Sikdar. *Efficient Exact Algorithms Through Enumerating Maximal Independent Sets and Other Techniques.* Theory Comput. Systems Vol. 41, pp. 563-587, 2007.
16. I. Razgon and B. O'Sullivan. *Almost 2-SAT is Fixed-Parameter Tractable.* In Proc. of ICALP 2008, Springer LNCS Vol. 5125, pp. 551-562.
17. B. Reed, K. Smith and A. Vetta. *Finding Odd Cycle Transversals.* Operations Research Letters Vol. 32, pp. 299-301, 2004.
18. F. Sterboul. *A Characterization of Graphs in which the Transversal Number Equals the Matching Number.* Jour. of Comb. Theory, Ser. B Vol. 27, pp. 228-229, 1979.