Are there any Good Digraph Width Measures?

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Outline

Measuring the Width of a Graph

Measures for Undirected Graphs

Treewidth [Robertson and Seymour] - very successful.

- FPT algorithms for many problems (MSO_2);
- nice closure properties;
- graphs of small treewidth have a rich structure.

Cliquewidth/Rankwidth [Courcelle and Olariu/ Oum and Seymour].

- again, FPT or XP algorithms for many problems (including all of MSO_1);
- not subgraph or minor closed.

Measuring the Width of a Graph

Width Measures for Directed Graphs?

Directed Treewidth [Johnson, Robertson, Seymour and Thomas].

- XP-algorithms for Hamiltonian Path and k-Path problems;
- technically difficult and not many efficient algorithms . . .

Recent Additions

- DAG width [Obdržálek];
- Kelly width [Hunter and Kreutzer].
- Directed Cliquewidth [Courcelle and Olariu].
- Birankwidth [Kanté].
- Kenny width.
- DAG depth.
- DFVS number.

Structural Properties of Digraph Width Measures

Very Good: DAG width, Kelly width, DAG-depth.

- nice cops-and-robber game characterizations;
- monotone under taking subgraphs.

Good: Directed treewidth, Kenny width, DFVS number.

- no game characterization but monotone under taking subgraphs.

Bad: Directed cliquewidth and Birankwidth.

- not monotone under taking subgraphs (a bi-oriented clique has small width but its subgraphs can have much larger width);
- but closed under vertex minors.

Algorithmic Usefulness

Very Good: Directed Cliquewidth and Birankwidth.

- all MSO_1 problems have FPT algorithms;
- many other problems have XP-algorithms.

Bad: All other measures!

Desirable Properties of a Digraph Width Measure

- Algorithmic usefulness many problems can be solved on digraphs of small width;
- Different from treewidth: otherwise, simply use the treewidth of the underlying undirected graph;
- Nice structural properties / a cops-and-robber game characterization.

We show that no digraph width measure satisfies all the above properties!

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Formalizing the Conditions

Algorithmic Usefulness

Definition

A digraph width measure is powerful if all problems in MSO_1 admit XP algorithms with the width as parameter.

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Being Different from Treewidth

Definition

A digraph width measure δ is treewidth-bounding if for all digraphs with width at most k, the undirected treewidth is at most $b(k)$.

We want digraph width measures to be *not* treewidth-bounding.

- class of digraphs with width at most c (constant) have arbitrary high undirected treewidth.

Formalizing the Conditions

Having nice structural properties/ cops-and-robber game characterization

Observation

- In most versions of cops-and-robber games, shrinking a (directed) path does not help the robber.
- Width measures based on cops-and-robber games are closed under some form of (directed) topological minor.

Formalizing the Conditions

When is a width measure cops-and-robber games based?

- when it is closed under directed topological minors.

Definition (Informal)

A digraph H is a directed topological minor of a digraph D, if H can be obtained by contracting certain arcs in a subdigraph of D.

Which arcs can be contracted?

- Arcs whose contraction does not create new dipaths between large degree vertices.

Formalizing the Conditions

Directed Topological Minors: Contractible Arcs

- let V_3 be the set of vertices with at least three neighbors.
- arc $\vec{a}$ is contractible if
  - not both end-points of $\vec{a}$ are in V_3;
  - contracting $\vec{a}$ does not create new dipaths between vertices of V_3.

Contractible Arcs: An Example

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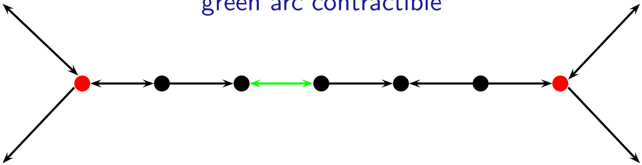
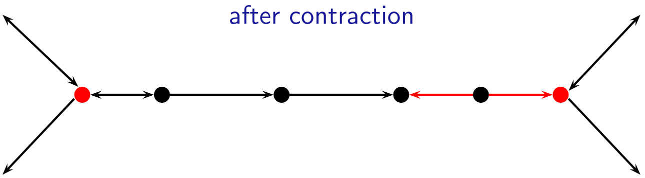
green arc contractible

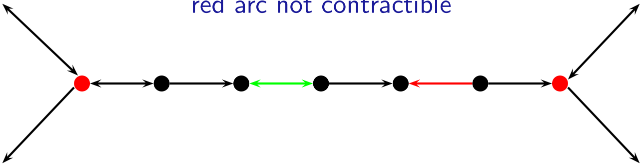


Contractible Arcs: An Example



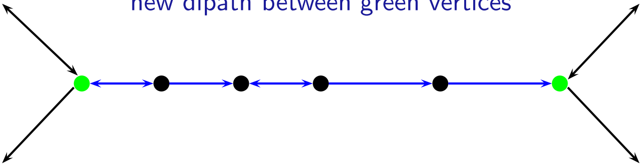
Contractible Arcs: An Example

red arc not contractible



Contractible Arcs: An Example

new dipath between green vertices



Outline

First Statement

Finally we need a technical property.

Efficient orientability tells us how to efficiently orient the edges of a given undirected graph to obtain a digraph of minimum width.

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Efficient orientability tells us how to efficiently orient the edges of a given undirected graph to obtain a digraph of minimum width.

Definition

A digraph width measure δ is called efficiently orientable if there exist functions $h\colon \mathbb{N} \to \mathbb{N}$ and $r : \mathscr{G} \to \mathscr{D}$, such that

1. *the function r can be computed in time polynomial in the input graph;*
2. *for every graph G, $r(G)$ is an orientation of G; and,*
3. $\delta(r(G)) \leq h(\min\{\delta(D)\colon U(D) = G\})$.

The Main Theorem

Theorem

Let δ be a digraph width measure such that

1. *δ is not treewidth-bounding;*
2. *δ is monotone under taking directed topological minors;*
3. *δ is efficiently orientable.*

Then $P = NP$, or δ is not powerful.

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Proof.

- There exists a constant $c \in \mathbb{N}$ such that $U(\delta(D) \leq c)$ contains undirected graphs of arbitrarily large treewidth.
- Every planar graph is a minor of some graph in $U(\delta(D) \leq c)$.
- Every $\{1, 3\}$-planar graph is a topological minor of some graph in $U(\delta(D) \leq c)$.

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Proof. Hence $U(\delta(D) \leq c)$ contains small subdivisions of every $\{1, 3\}$-regular planar undirected graph.

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Proof. Given a $\{1, 3\}$-planar graph, efficiently find an orientation of width at most $h(c)$.

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Proof. There exists NP-complete problems on $\{1, 3\}$-planar graphs that are MSO_1-expressible.

Strengthening the Result?

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Can we strengthen the result by requiring closure under subdigraphs?

No!

Strengthening the Result?

Theorem

There exists a digraph width measure δ with these properties:

1. *δ is not treewidth-bounding;*
2. *δ is monotone under taking subdigraphs;*
3. *δ is efficiently orientable;*
4. *δ is powerful.*

Proof.

By a padding argument and Courcelle's Theorem for treewidth. □

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Can the result be strengthened at all?

Doesn't seem so.

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Why do we require efficient orientability?

Strengthening the Result?

Efficient orientability prevents a width measure from keeping excessive information in the orientation of the arcs.

Strengthening the Result?

Theorem

There exists a digraph width measure δ with these properties:

1. *δ is not treewidth-bounding;*
2. *δ is monotone under taking directed topological minors;*
3. *for every $k \geq 1$, for every digraph D with $\delta(D) \leq k$, it can be decided in time $\mathcal{O}(3^k \cdot n^2)$ whether $U(D)$ is 3-colourable.*

Proof.

Arcs directions are used to encode a 3-coloring:

- sources form one color class;
- sinks form another color class;
- the rest form the third color class.

□

Outline

On Being Powerful

For digraph width measures that are

1. not treewidth-bounding,
2. efficiently orientable,

we have identified a threshold with respect to being powerful:

monotone under subgraphs	$\longrightarrow$	powerful;
monotone under directed topological minors	$\longrightarrow$	not powerful.

On Efficient Orientability

We have given evidence that this property is necessary for proving our result.

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Question

Is there digraph width measure that is

1. *not treewidth-bounding;*
2. *monotone under taking directed topological minors; and*
3. *not efficiently orientable; and,*
4. *powerful?*

On Directed Topological Minors

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Theorem

There exists a digraph H such that, given a digraph G, deciding whether H is directed topological minor of G is NP-complete.

Thank You!