Linear Kernels for Chordal Deletion Problems on Sparse Graphs

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Outline

Basic Definitions

Parameterized Problems

Decision problems with two components (x, k), where k is the **parameter**.

Examples

- Vertex Cover: given (G, k), does G have a vertex cover of size at most k?
- Dominating Set: given (G, k), does G have a dominating set of size at most k?
- Longest Common Subsequence: given a sequences $S_1, \ldots, S_r$ from some fixed alphabet and integer k, does the longest common subsequence have length at least k?

Fixed-Parameter Tractability

Running times are **measured wrt both x and k**.

Definition

A parameterized problem is **fixed-parameter tractable** if there is an algorithm with running time $O(f(k) \cdot |x|^c)$, where f is a function of k alone and c is a constant.

A closely related concept: **kernelization algorithm**.

Kernelization and Fixed-Parameter Tractability

Definition

A **kernelization algorithm** for a parameterized problem is **polynomial-time many-one reduction** mapping an instance (x, k) to (x', k') s.t.

- (x, k) is a yes-instance iff (x', k') is a yes-instance;
- $|x'|, k' \leq f(k)$, for some function f.

The function f is called the **size of the kernel**.

Folklore

A problem is fixed-parameter tractable (FPT) iff it has a kernelization algorithm.

Kernel Sizes

The kernel size obtained from a fixed-parameter algorithm is **usually exponential or worse**.

Goal

To obtain polynomial (or even better, linear) kernels.

Basic Technique

- devise **reduction rules** that preserve **equivalence** of instances;
- when reduction rules cannot be applied anymore, show that the resulting instance has small size.

Outline

Hereditary Properties . . .

. . . are a class of graphs closed under vertex deletion.

- e.g. acyclic, bipartite, chordal, planar, bounded-degree, degenerate, interval, proper interval.

Observation

A class is hereditary iff it has a forbidden set characterization.

Examples

- Acyclic: all cycles.
- Bipartite: all odd cycles.
- Chordal: all holes (chordless cycles of length at least four).

Not always easy to obtain the forbidden set (try Interval, Planar).

Decision Problems Associated with Hereditary Properties

Given a hereditary property Π,

Definition ($\Pi(i, j, k)$-Graph Modification)

Given a graph G and integers i, j, k, can one delete at most i vertices, at most j edges and add at most k edges s.t. the resulting graph satisfies Π?

Definition (Π-Induced Subgraph)

Given a graph G and an integer k, does G have a vertex-induced subgraph with at least k vertices that satisfies Π?

- NP-complete [Papadimitriou and Yannakakis, 1978].
- Parameterized Complexity?

Π-Induced Subgraph

A complete characterization wrt inclusion in FPT is known.

Theorem (Khot and Raman, 2001)

If Π contains all independent sets and all cliques, then the Π-Induced Subgraph problem is in FPT. Else it is $W[1]$-complete.

For hereditary properties Π on directed graphs . . .

Theorem (Raman and S., 2006)

The Π-Induced Subgraph problem is in FPT if Π contains all independent sets, all acyclic tournaments and all complete symmetric digraphs. Else it is $W[1]$-complete.

Open. A complete characterization as to when these problems have a polynomial kernel.

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Π-Graph Modification

For properties with a finite forbidden set

Theorem (Cai, 1996)

If Π is hereditary and a has a finite forbidden set then the $\Pi(i, j, k)$-Graph Modification problem is in FPT.

Polynomial Kernel: Reduce to d-Hitting Set.

Properties with an infinite forbidden set:

- Feedback Vertex Set: in FPT; quadratic kernel.
- Odd Cycle Transversal: in FPT; randomized poly kernel.
- Chordal Vertex Deletion: in FPT; poly kernel?
- Chordal Completion: in FPT; $O(k^2)$-vertex kernel.
- Proper Interval Completion: in FPT; $O(k^5)$-vertex kernel.

The Π-Vertex Deletion Problem

A restriction of the Π-Graph Modification problem.

Definition

Is there a vertex-set S of size at most k whose deletion results in a graph with property Π?

Special cases

- Feedback Vertex Set: in FPT; quadratic kernel.
- Odd Cycle Transversal: in FPT; randomized poly kernel.
- Chordal Vertex Deletion: in FPT; poly kernel?
- Wheel-Free Deletion: W[2]-complete.
- Directed Feedback Vertex Set: in FPT; poly kernel?

Outline

Π-Vertex Deletion: Main Results

We only consider hereditary properties whose forbidden sets have **connected** graphs.

Theorem

Let Π be a hereditary property. If (G, k) is a yes-instance of Π-Vertex Deletion,

- *then there exists $S \subseteq V(G)$ of size at most k s.t. $\mathbf{tw}(G \setminus S)$ is bounded.*

Then the Π-Vertex Deletion problem on H-topological-minor-free graphs admits a linear kernel.

Π-Vertex Deletion: Main Results . . .

Special Case

For hereditary properties that contain all **holes**, the condition

$$\textbf{tw}\,(G \setminus S) < \textbf{some constant}$$

holds if G is ***H*-topological-minor-free**.

- Chordal Vertex Deletion.
- Interval Vertex Deletion.

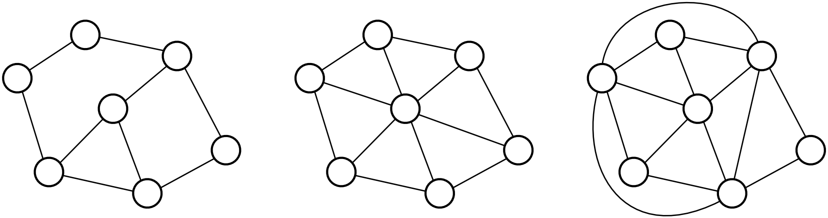
Corollary

Fix a graph H. If Π is a hereditary property whose forbidden set contains all holes, then the Π-Vertex Deletion problem admits a linear kernel in H-topological-minor-free graphs.

Holes and Chordal Graphs

Definition

A **hole** is an induced cycle of length at least four. A graph is **chordal** if it does not contain any holes.



Some Properties of Chordal Graphs

A vertex is **simplicial** if its neighbourhood induces a clique.

Property

A chordal graph is either a clique or has at least two non-adjacent simplicial vertices.

Perfect Elimination Order

An ordering of vertices s.t. for each vertex v, v **and all neighbors occurring after it induce a clique**.

Property

A graph is chordal iff it has a perfect elimination order.

Tree-Decompositions of Chordal Graphs

Lemma

An optimal tree-decomposition of a chordal graph can be obtained in poly time.

Proof Idea.

- Identify a simplicial vertex u; create a bag containing $N[u]$; delete u.
- Repeat until there are no vertices are left.

The bags can be strung together to a valid tree-decomposition.

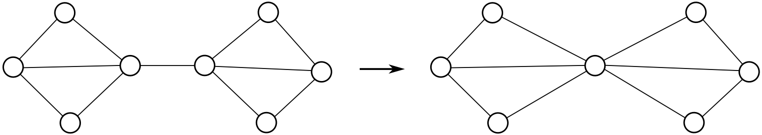
- Each bag is a maximal clique.
- **Bounded clique-size implies bounded treewidth**.

Chordal graphs consist of **"overlapping maximal cliques in a tree-like structure"**.

Minors and Topological Minors

Definition

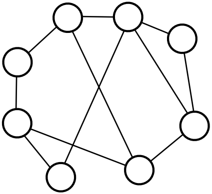
A graph H is a **minor** of G, if it can be obtained from a subgraph of G by a sequence of edge contractions.



Minors and Topological Minors . . .

Definition

A graph H is a **topological minor** of G, if it can be obtained from a subgraph of G by contracting edges $e = \{x, y\}$ s.t. $\deg(x) \leq 2$.



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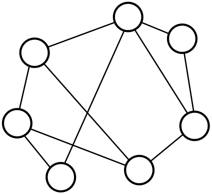

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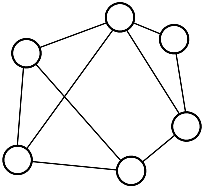

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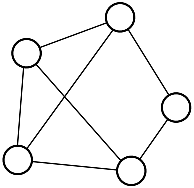

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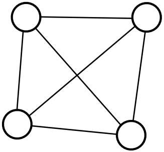


Implications of H-(Topological)-Minor-Freeness

Fix H and let $r := |V(H)|$.

Bounded Average Degree

- $cr\sqrt{\log r}$: minor-free [Kostochka, 1984].
- $c'r^2$: topological-minor-free [Komlós and Szemerédi, 1996].

Bounded Clique Size

- no cliques with $\geq r$ vertices.

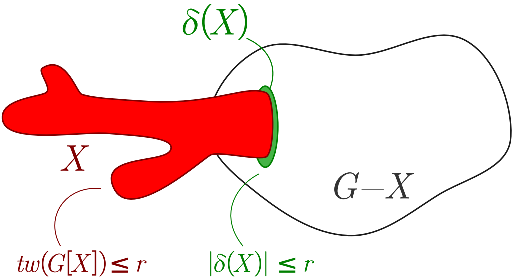
Bounded Number of Cliques

- d-degenerate implies at most $2^d \cdot n$ cliques [Wood, 2007].
- minor-free: $2^{cr \log r} \cdot n$.
- topological-minor-free: $2^{c'r^2} \cdot n$.

The Protrusion Machinery

Definition

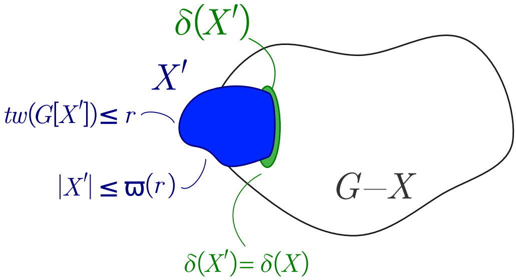
A **protrusion** is a subgraph of bounded treewidth that is connected to the rest of the graph by a small separator.



Reductions based on Protrusions

Reduction Rule

If X is a protrusion whose size is larger than some constant (that depends only on the problem), replace it with a smaller protrusion X' s.t. the solution remains the "same".



Kernels based on Protrusion Reductions

Either by some combinatorial result or simply due to the problem specification:

> *If (G, k) is a yes-instance and G is large, then G has a large protrusion.*

Hence

> *If (G, k) is a yes-instance and G has no large protrusions, then G must be small.*

Outline

Main Result

Theorem

Fix H. If Π is a hereditary property whose forbidden set contains all holes, then the Π-Vertex Deletion problem admits a linear kernel in H-topological-minor-free graphs.

Reduction Rule

Let $r := |V(H)|$. Replace all $3r$-protrusions by equivalent ones of size at most $\varpi(3r)$.

How do we find such protrusions? Use brute-force to find separators of size at most $3r$. **Not practical!**

This is the only reduction rule we use.

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Stage I: Partitioning into Components

- (G, k): a yes-instance (G is H-topological-minor-free).
- If $S \subseteq V(G)$ is a solution then $\mathbf{tw}(G \setminus S) \leq r := |V(H)|$.

Stage I: Partitioning into Components

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Connected components in $G \setminus S$

- $\mathcal{C}_1$: adjacent to **at most** $r - 1$ **vertices of** S;
- $\mathcal{C}_2$: adjacent to $\geq r$ **vertices of** S.

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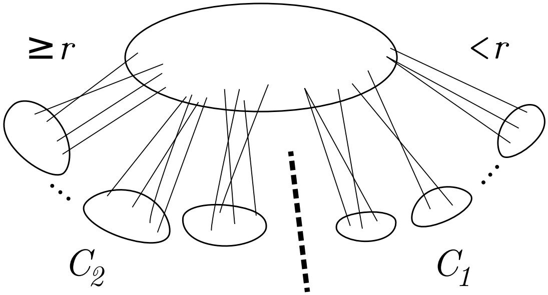

Stage II: Bounding the Size $\mathcal{C}_1$

$(r-1)$-Protrusions and the Effect of Reductions

- Components in $\mathcal{C}_1$ connected exclusively to some $X \subseteq S$.
- #vertices in all components connected to $X \leq \varpi(r-1)$.

Constructing a Topological Minor $\mathscr{S}$

- Delete $\mathcal{C}_2$; "contract" $C \in \mathcal{C}_1$ to edges in S without creating multiple edges; delete remaining components in $\mathcal{C}_1$.
- $\mathscr{S} \preceq_{top} G$ and hence is H-topological-minor-free.

Bounding the Size

- $\mathscr{S}$ contains at most $O(k)$ **edges** and $O(k)$ **cliques**.
- For each clique in $\mathscr{S}$, **#adjacent vertices in $\mathcal{C}_1$ is** $\leq \varpi(r-1)$.
- Hence **total number of vertices in $\mathcal{C}_1$ is** $O(k)$.

Stage III: Bounding the Size of $\mathcal{C}_2$

Recall: Components in $\mathcal{C}_2$ see at least r vertices in S.

Bounding the Number of Components in $\mathcal{C}_2$

Lemma

Let $V_1, \ldots, V_p$ are vertex-disjoint sets in $G \setminus S$ s.t. for $1 \leq i \leq p$,

- *$G[V_i]$ is connected;*
- *$G[V_i]$ "sees" at least r vertices in S.*

Then $p = O(k)$.

Stage III: Bounding the Size of $\mathcal{C}_2$ . . .

Task

Decompose the components in $\mathcal{C}_2$ into **connected** pieces s.t.

- each piece has **size roughly** $\varpi(3r)$;
- each piece **"sees" at least** r **vertices in** S.

By the previous lemma,

- there can be $O(k)$ **such pieces**;
- #vertices in $\mathcal{C}_2$ is at most $O(k \cdot \varpi(3r)) = O(k)$.
- This is technical and we won't present it here!

Outline

Extensions

Theorem

Let Π be a hereditary property. If (G, k) is a yes-instance of Π-Vertex Deletion,

- *then there exists $S \subseteq V(G)$ of size at most k s.t. $\mathbf{tw}(G \setminus S)$ is bounded.*

Then the Π-Vertex Deletion problem on H-topological-minor-free graphs admits a linear kernel.

- Chordal Vertex Deletion, Interval Vertex Deletion, Proper Interval Vertex Deletion.
- Feedback Vertex Set.

Other Issues

- The protrusion reduction takes polynomial time, but can hardly be called efficient.
- Is there a simpler algorithm based on less "powerful" reduction rules?
- A characterization of hereditary properties with infinite forbidden sets into FPT/W-hard.

Thank You!