A Faster Parameterized Algorithm for Treedepth

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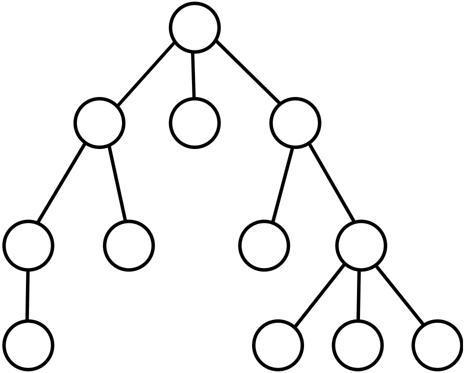
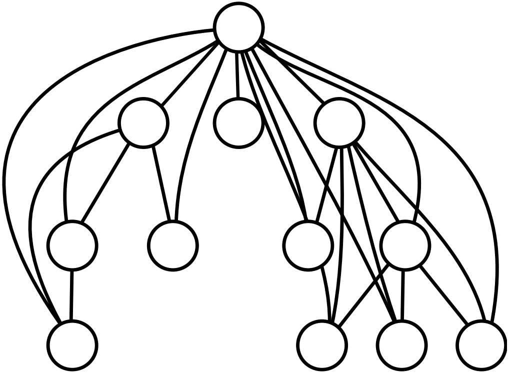
Somnath Sikdar

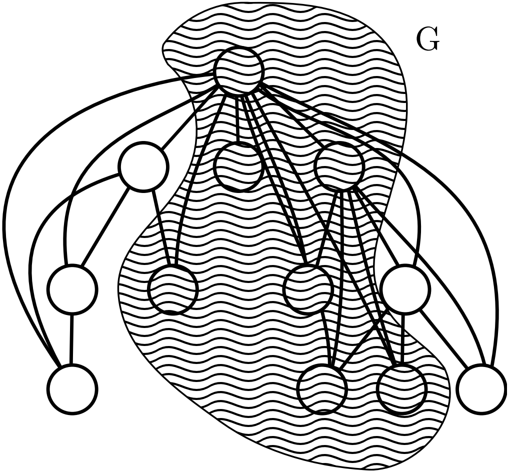
RWTH Aachen University

July 11, 2014

Treedepth is a width measure.



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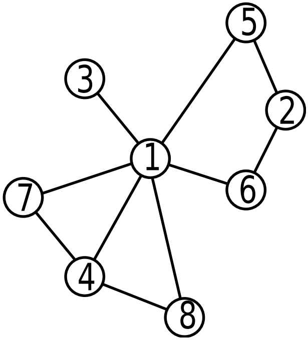
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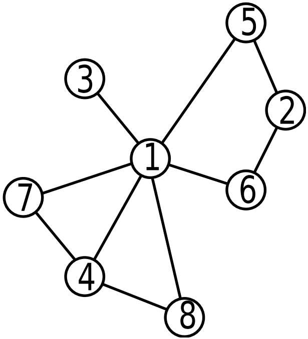
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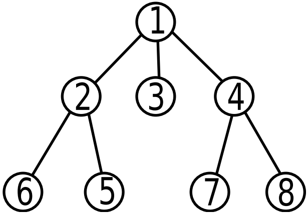
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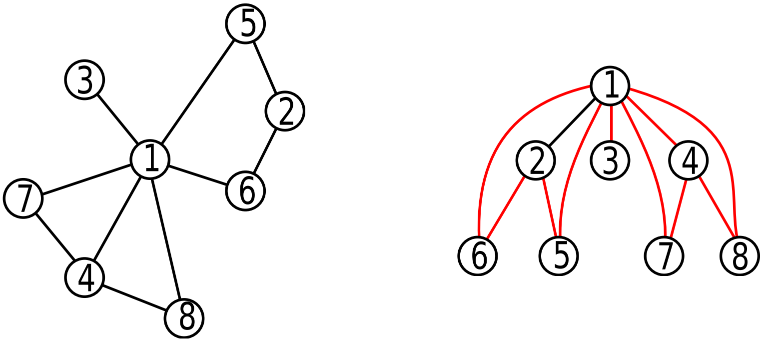
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Definition (Treedepth decomposition)

A *treedepth decomposition* of a graph G is a rooted forest F such that $V(G) \subseteq V(F)$ and $E(G) \subseteq E(clos(F))$.

Definition (Treedepth)

The *treedepth* $\mathbf{td}(G)$ of a graph G is the minimum height of any treedepth decomposition of G.

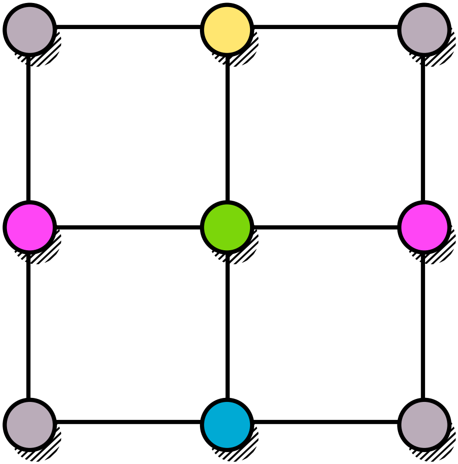
A strange width measure...

> "So many choices"
>
> —Dr. Dre

A graph G has *treedepth* at most t if

- G is a subgraph the closure of a tree (forest) of height $\leq t$
- G has a *centered coloring* with t colors
- G has a *ranked coloring* with t colors
- G is the subgraph of a *trivially perfect graph* with clique size $\leq t$

Centered Coloring



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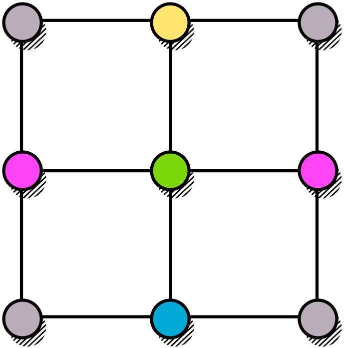

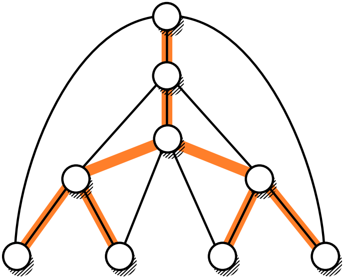

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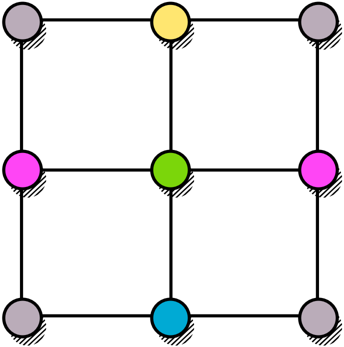
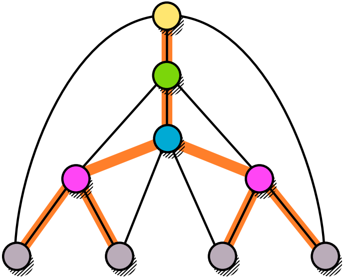
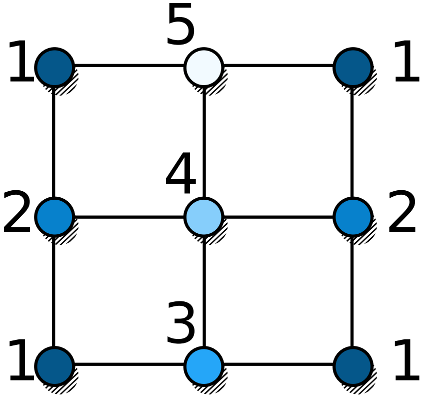
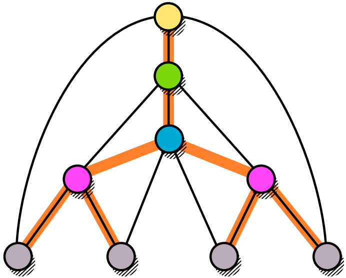




Ranked Coloring





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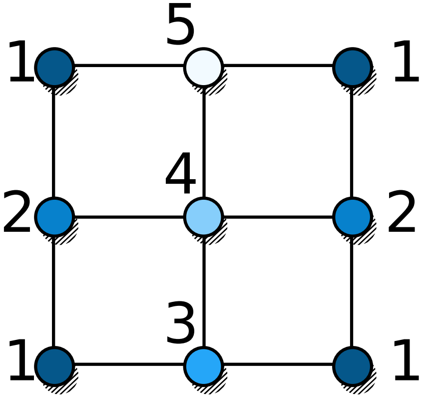

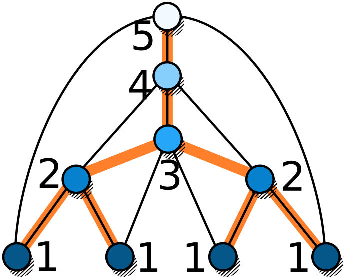

Trivially Perfect Graphs

G is the subgraph of a *trivially perfect graph* with clique size at most t.

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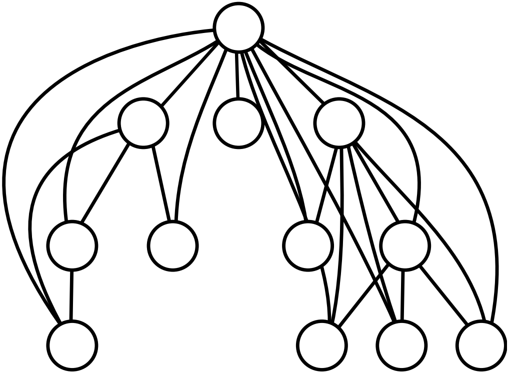

Arises again and again

Introduced as...

- *minimum elimination tree* by Pothen [1988]
- *ordered coloring* by Katchalski et al. [1995]
- *vertex ranking* by Bodlaender et al. [1998]
- **again** as treedepth by Nešetřil and Ossona de Mendez [2008]

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Related to...

- layouting of VLSI chips
- star height of regular languages
- characterizing bounded expansion graph classes
- counting subgraphs [New results coming]

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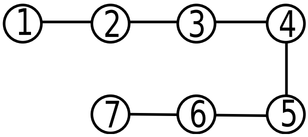
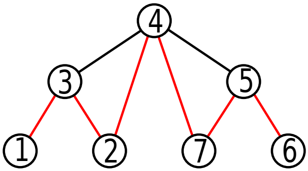
Personal opinion: Treedepth is the most useful definition.





Treedepth $t \rightarrow$ Maximal path length $2^t - 1$.

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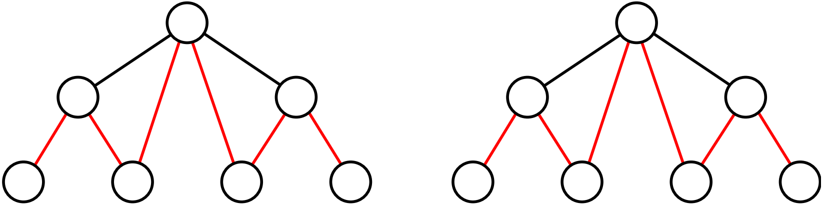

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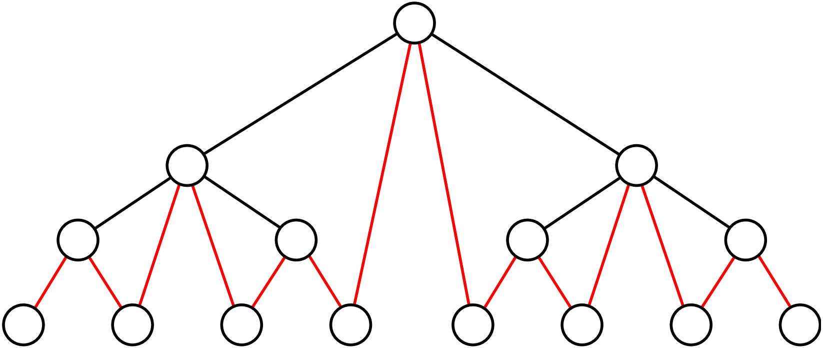
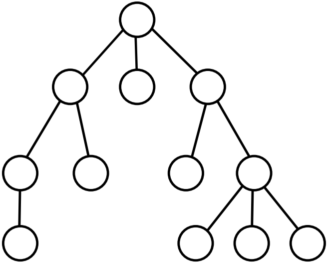


A DFS is a Treedepth decomposition



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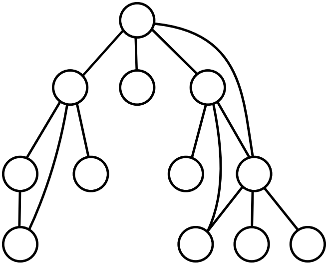

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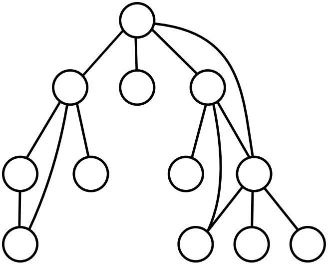
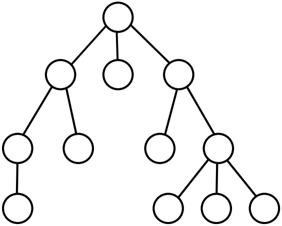


Treedepth $t \Rightarrow$ Maximal path length $2^t - 1 \Rightarrow 2^t$-approximation

Treedepth to pathwidth



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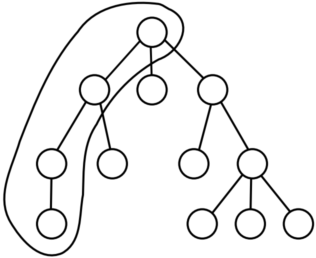

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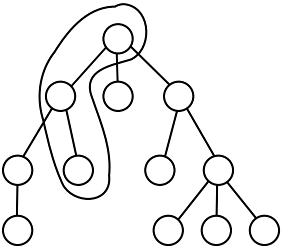

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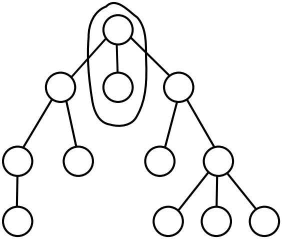

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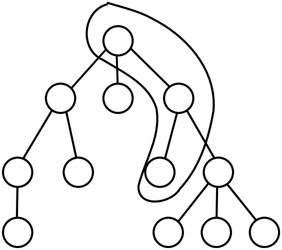

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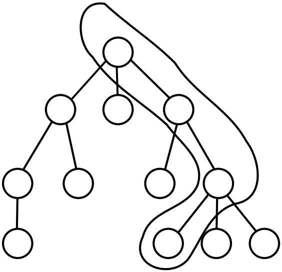

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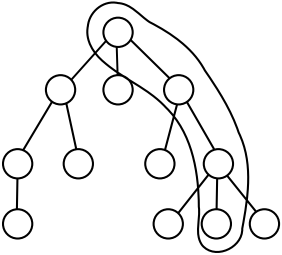

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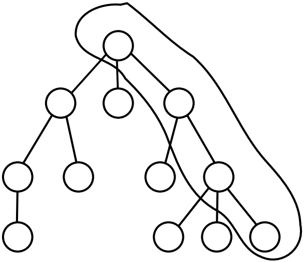

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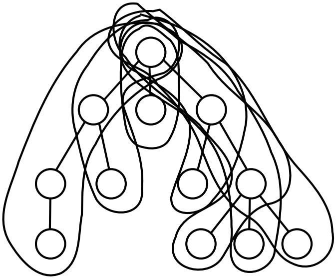
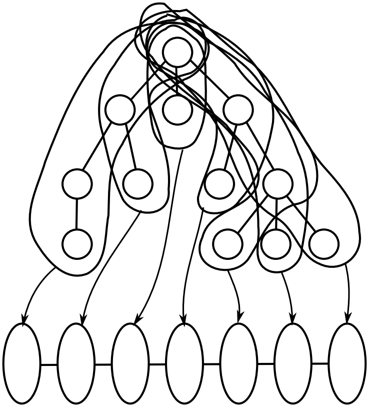
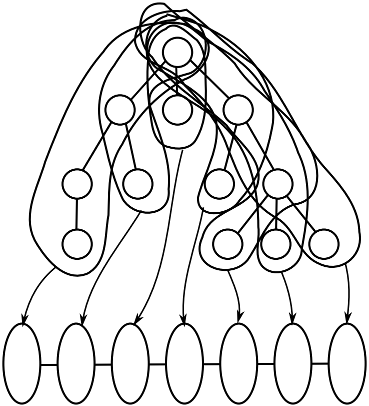


Treedepth to pathwidth

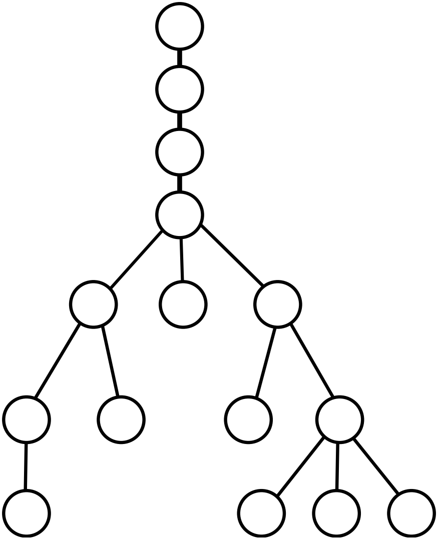


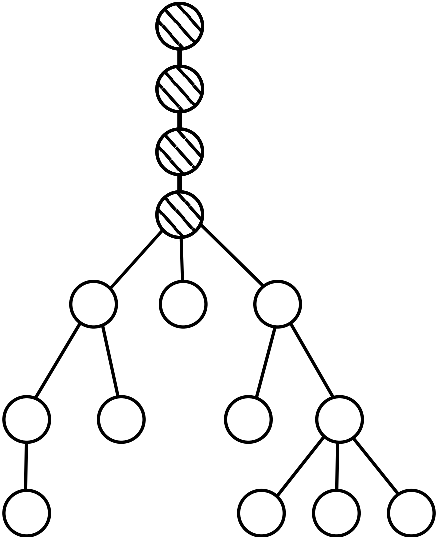
Treedepth to pathwidth

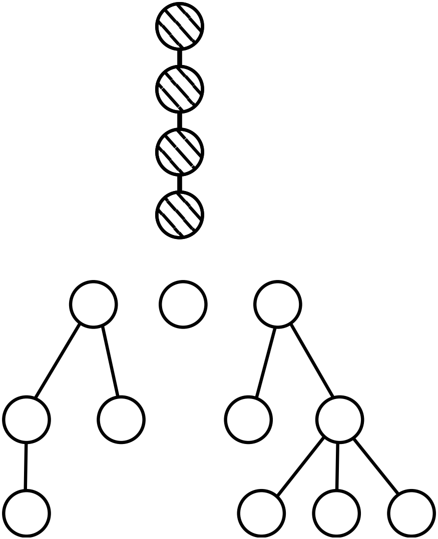


$$\mathbf{tw}(G) \leq \mathbf{pw}(G) \leq \mathbf{td}(G) - 1$$

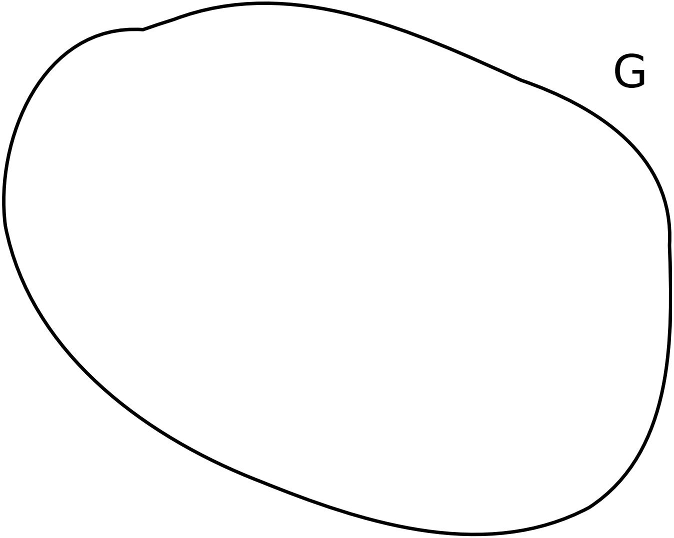
Treedepth $t \Rightarrow$ Path decomposition of width $2^t - 2$







Treedepth by bruteforce



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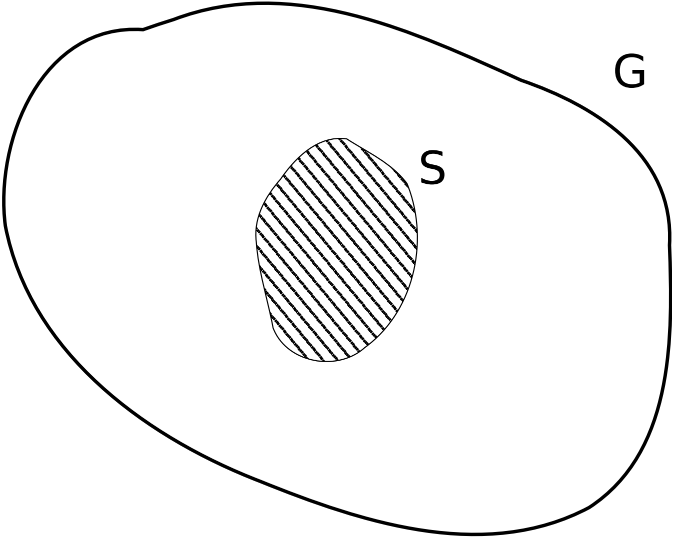

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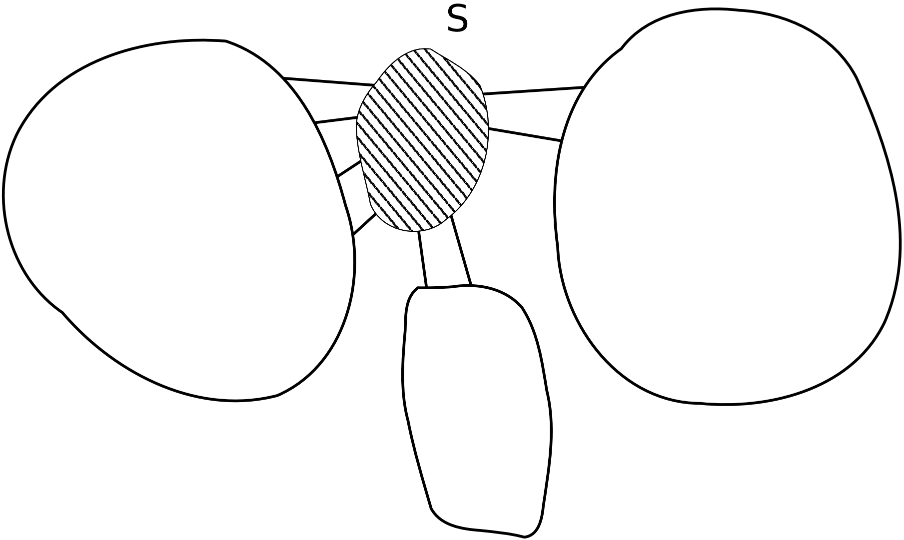

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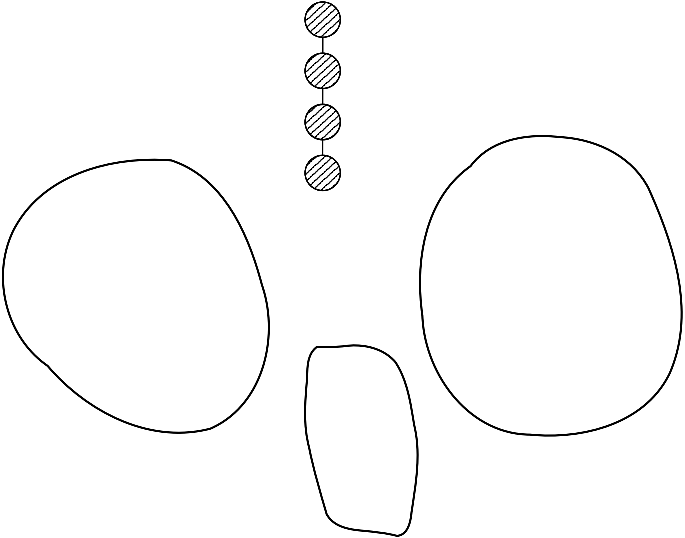

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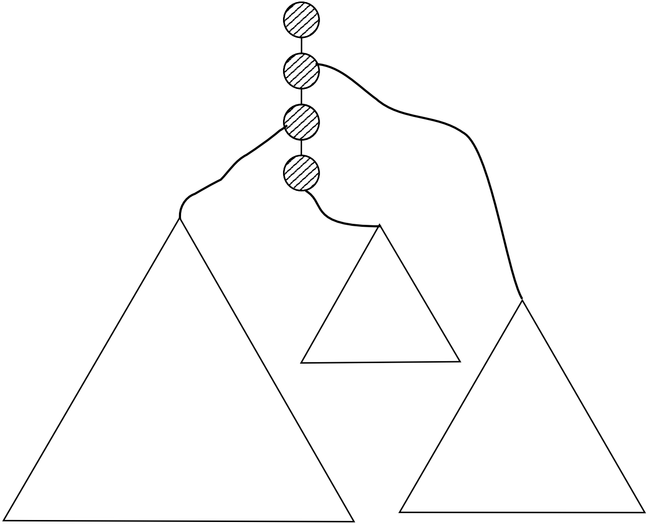

Parameterized algorithms

Open problem by Nešetřil and Ossona de Mendez [2012]

Is there a simple linear time algorithm to check $\mathbf{td}(G) \leq t$ for fixed t?

Parameterized algorithms

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Is there a simple linear time algorithm to check $\mathbf{td}(G) \leq t$ for fixed t?

- In $f(t) \cdot n^3$ time by Robertson and Seymour.
- $\mathbf{tw}(G) \leq \mathbf{td}(G) - 1 \Rightarrow$ By Courcelle's Theorem $2^{2^{2^{\cdot^{\cdot^{\cdot^{t}}}}}} \cdot n$.
- Algorithm by Bodlaender et. al. with running time $2^{O(w^2 t)} \cdot n^2$.

Our results:

- A (relatively) simple direct algorithm in time $2^{2^{O(t)}} \cdot n$.
- A fast algorithm in time $2^{O(t^2)} \cdot n$.

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- A (relatively) simple direct algorithm in time $2^{2^{O(t)}} \cdot n$.
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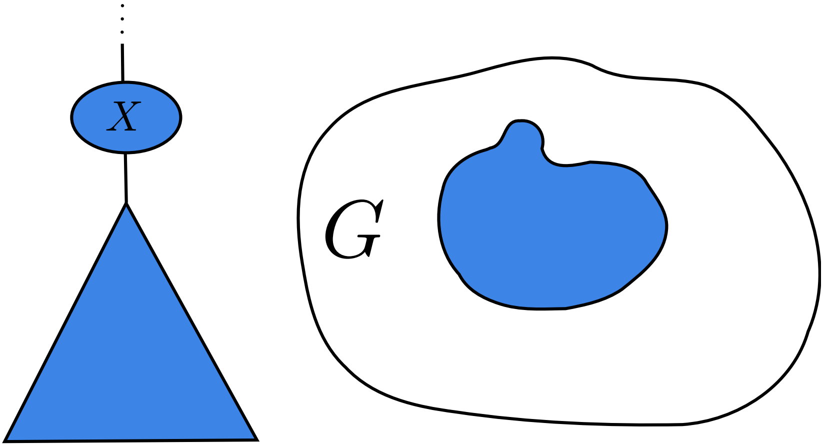
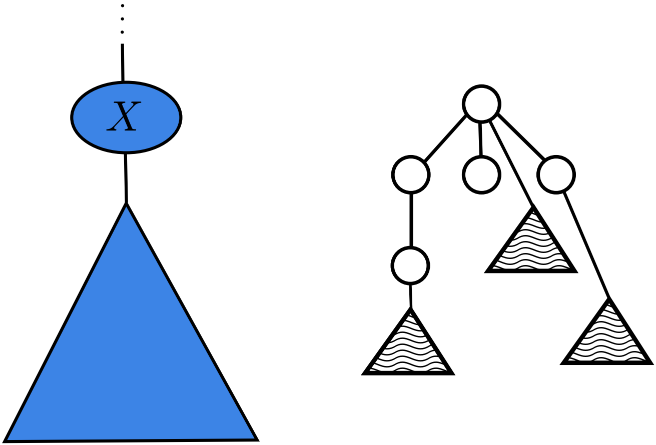
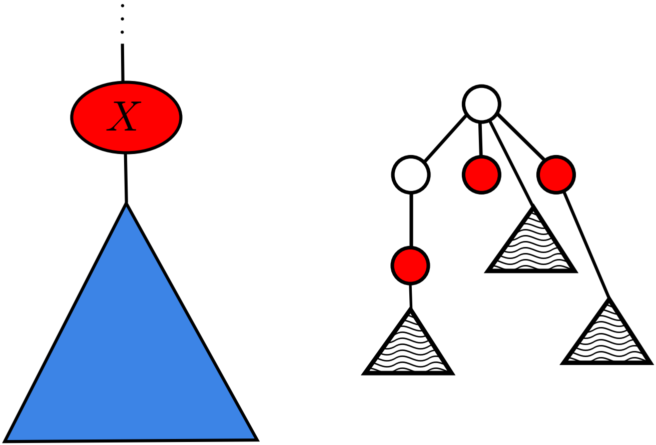
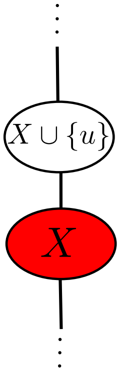
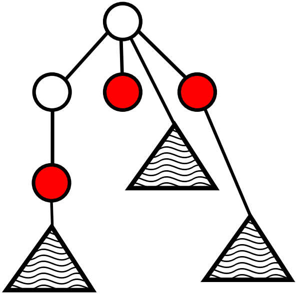
Both results follow from an algorithm on tree decompositions which runs in time $2^{O(wt)} \cdot n$.

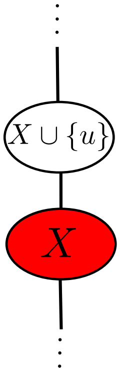

X
G

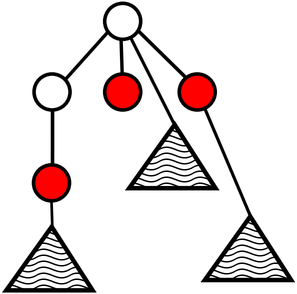

X


X

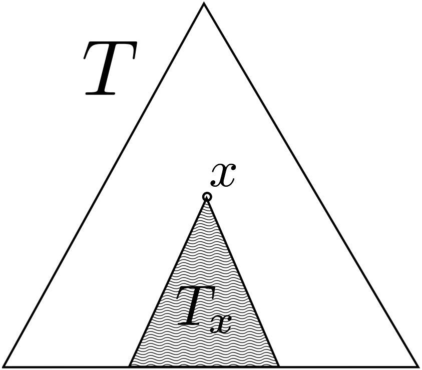

$X \cup \{u\}$
X

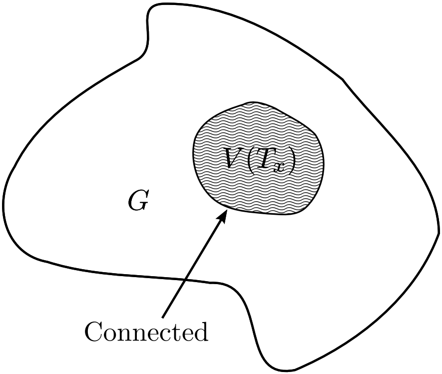


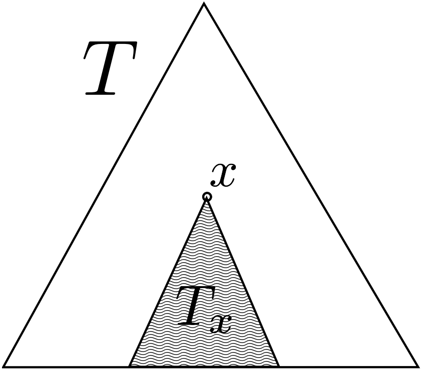


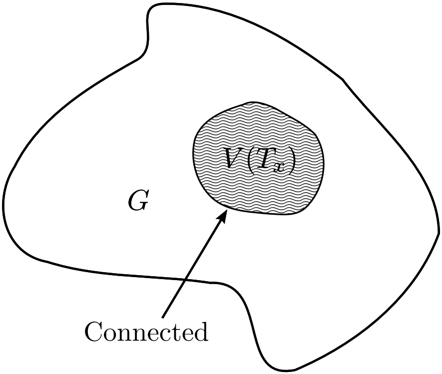


Where could the introduced node u be?


T
x
T_x

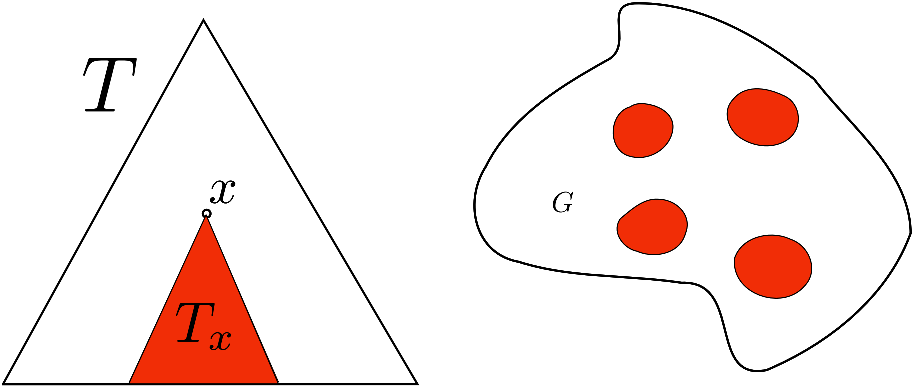

$V(T_x)$
G
Connected

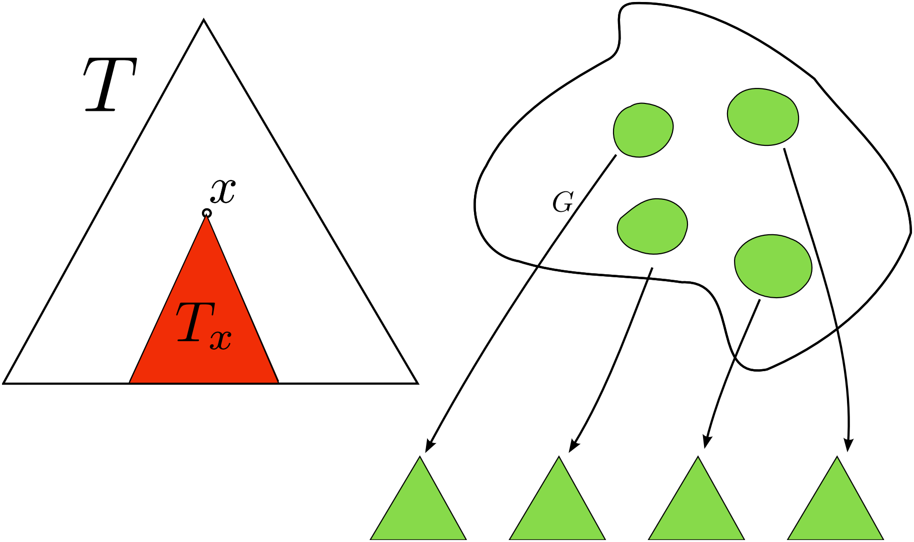


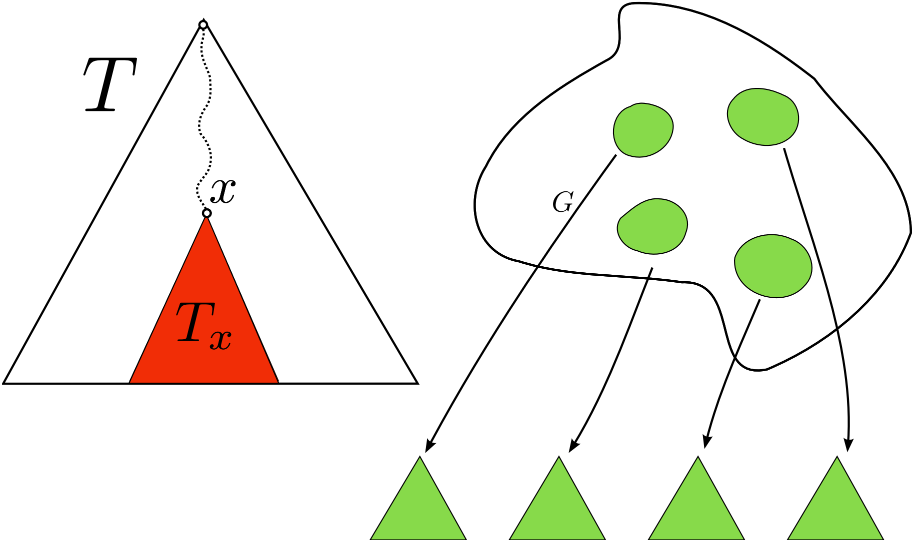


Definition (Nice treedepth decomposition)

We say that T is *nice* if for every vertex $x \in V(T)$, the subgraph of G induced by the vertices in T_x is connected.


T
x
T_x
G


T
x
T_x
G


T
x
T_x
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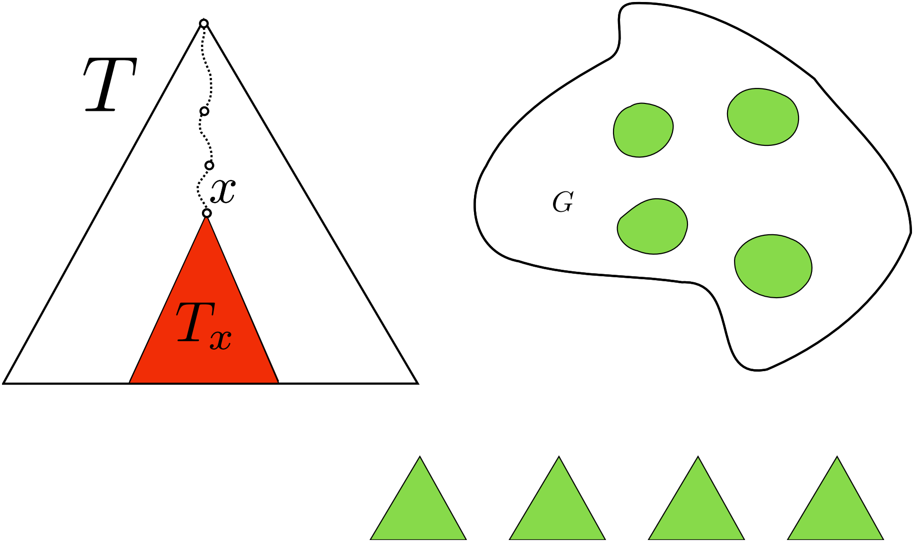
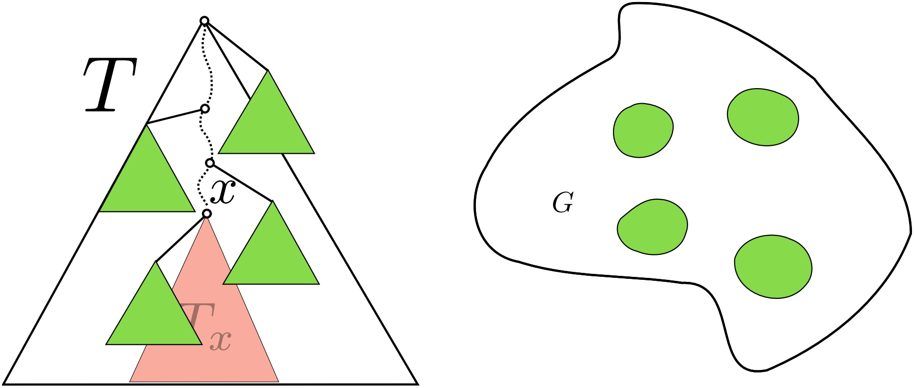
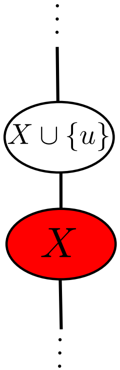
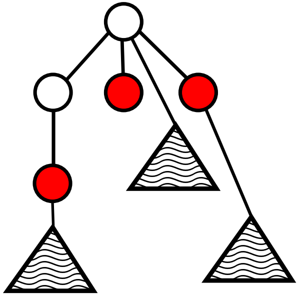

T
x
T_x
G


T
x
T_x
G

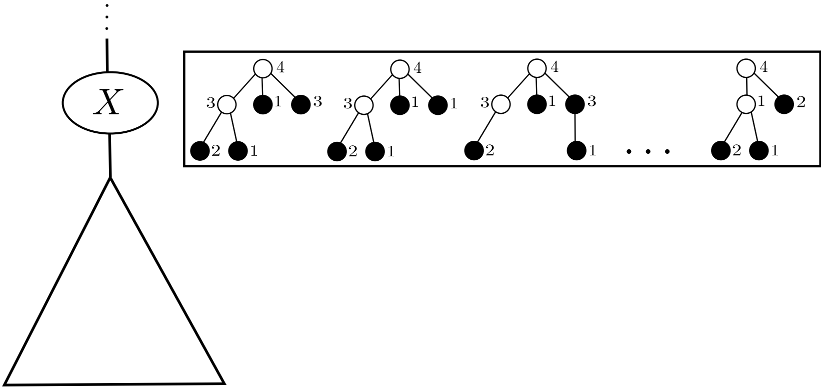
Lemma

For any graph there exists a treedepth decomposition of minimal depth which is nice.





Where could the introduced node u be?


X
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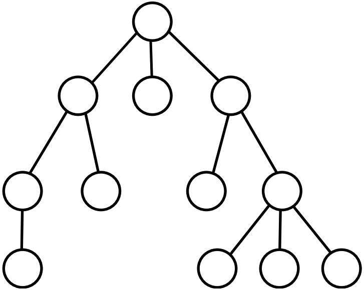

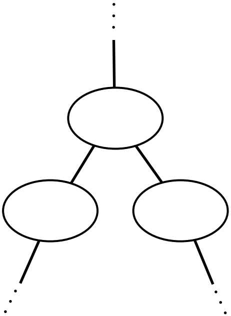

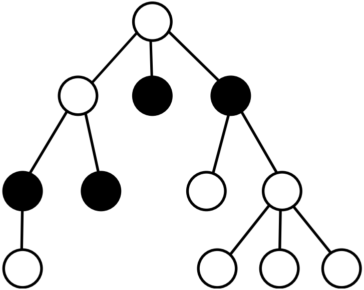

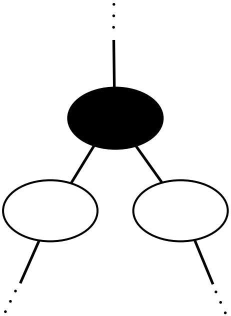

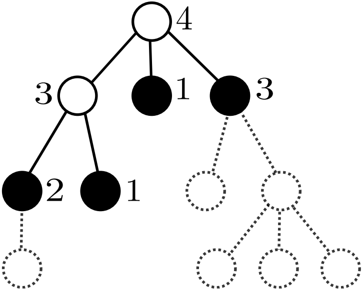
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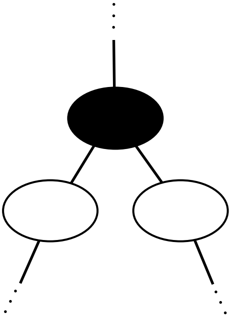

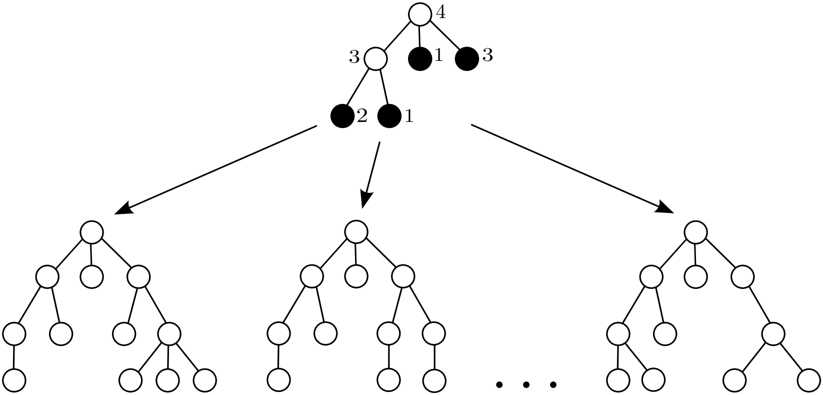
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Theorem

Given a graph G with n nodes and a tree decomposition of G of width w the treedepth t of G can be decided in time $2^{O(wt)} \cdot n$.

Simple algorithm

1. Depth-first-search to construct treedepth decomposition T.
2. If depth greater than $2^t - 1$ say NO.
3. Construct path decomposition $\mathcal{P}$ from T of width 2^t.
4. Run algorithm on $\mathcal{P}$.

Theorem

There is a (simple) algorithm to decide if a graph G with n nodes has treedepth t which runs in time $2^{2^{O(t)}} \cdot n$.

Fast algorithm

1. Use single exponential 5-approximation for treewidth[1].
2. Remember $\mathbf{tw}(G) \leq \mathbf{pw}(G) \leq \mathbf{td}(G) - 1$.
3. If width is greater than $5t$ say NO.
4. Else run algorithm on tree decomposition.

Theorem

There is a algorithm to decide if a graph G with n nodes has treedepth t which runs in time $2^{O(t^2)} \cdot n$.

[1] Very useful result by Hans Bodlaender, Pål G. Drange, Markus S. Dregi, Fedor V. Fomin, Daniel Lokshtanov and Michał Pilipczuk

Thank you for listening.

Questions?