Linear kernels for graphs excluding a topological minor

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Contents

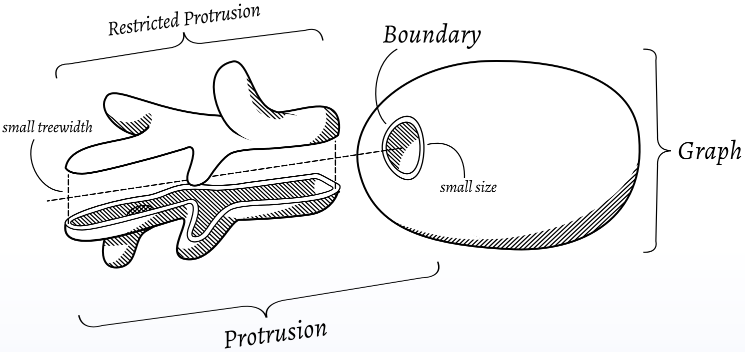
Linear kernels in sparse graphs

Overview

- Framework for planar graphs
  Guo and Niedermeier: *Linear problem kernels for NP-hard problems on planar graphs*
- Meta-result for graphs of bounded genus
  Bodlaender, Fomin, Lokshtanov, Penninkx, Saurabh and Thilikos: *(Meta) Kernelization*
- Meta-result for graphs excluding a fixed graph as a minor
  Fomin, Lokshtanov, Saurabh and Thilikos: *Bidimensionality and kernels*
- *Our contribution*: general result for graphs excluding a fixed graph as a *topological* minor

Reduction via protrusions

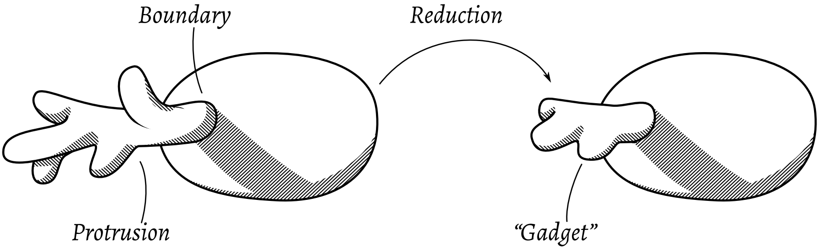
Protrusion anatomy



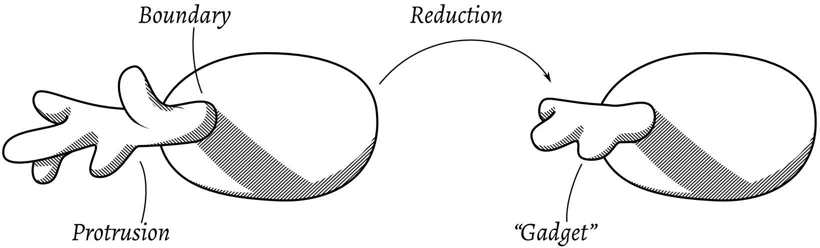
Definition

$X \subseteq V(G)$ is a t-*protrusion* if

1. $|\partial(X)| = |N(X) \setminus X| \leqslant t$ (small boundary)
2. $\mathbf{tw}(G[X]) \leqslant t$ (small treewidth)



We want to replace a large protrusion by something smaller.
Requires that the problem...



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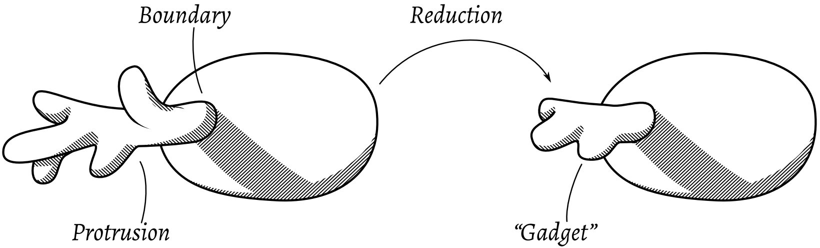

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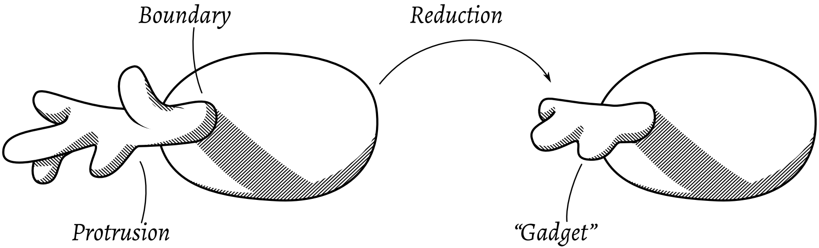

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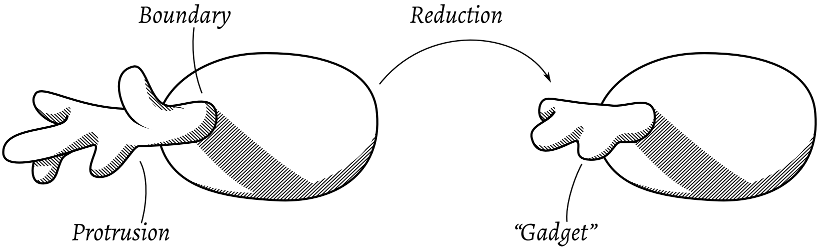

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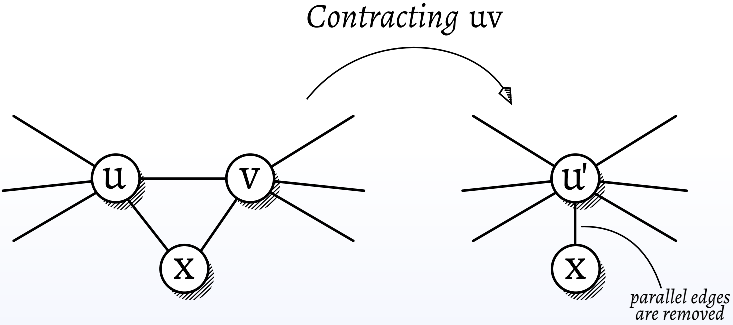
Note: the reduction can decrease the parameter.
This is the only reduction.

(Topological) Minors

Edge contraction



Graph relations

(now with contractions!)

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Relation	Operations
induced subgraph	delete vertices
subgraph	
topological minor	
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Graph relations

(now with contractions!)

Relation	Operations
induced subgraph	delete vertices
subgraph	delete vertices and edges
topological minor	delete vertices and edges, contract edges *incident to a degree-2 vertex*
minor	delete vertices and edges, contract edges

Properties of $\mathcal{H}$-topological-minor-free graphs

Let G be a graph excluding H as a topological minor.

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- Not interested in structure of H, but its size $r = |H|$

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1. $m \leqslant \frac{1}{2}\beta r^2 n$ (for some $\beta < 10$)

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2. #cliques $\leqslant 2^{\tau r \log r} n$ (for some $\tau < 4.51$)
3. Closed under taking topological minors

Our result and how it works

Requirements

(besides the ones mentioned before)

Definition (Treewidth bounding)

A parameterized graph problem Π is called *treewidth bounding* if for every $(G, k) \in \Pi$ it holds that there exists a set $S \subseteq V(G)$ such that

1. $|S| \leqslant ck$
2. $\mathbf{tw}(G - S) \leqslant t$

for constants c, t only depending on Π.

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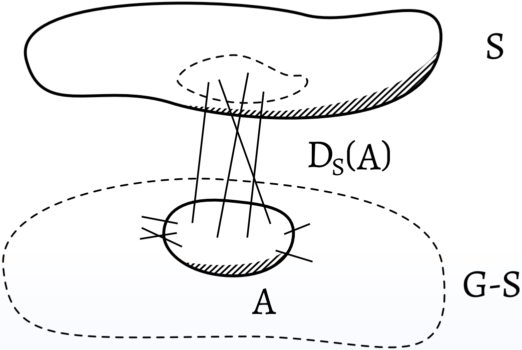
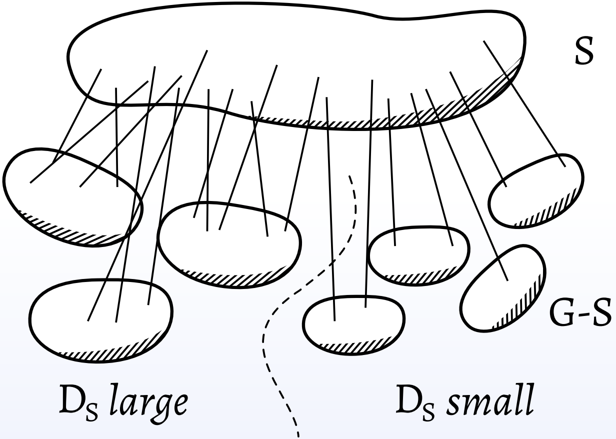
- Chordal Vertex Deletion in graphs with bounded clique-size

A little bit of notation

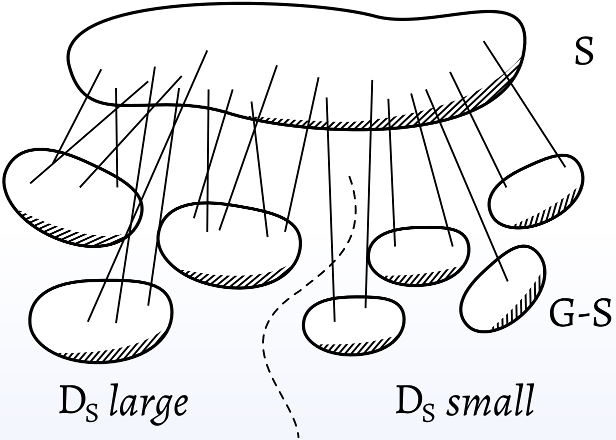


We write $D_S(A) = |\{u \in S \mid v \in A : uv \in E(G)\}|$ for the number of vertices in S that have neighbours in A (for disjoint sets S, A)

A decomposition

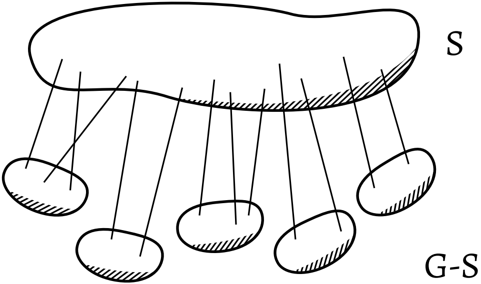


A decomposition



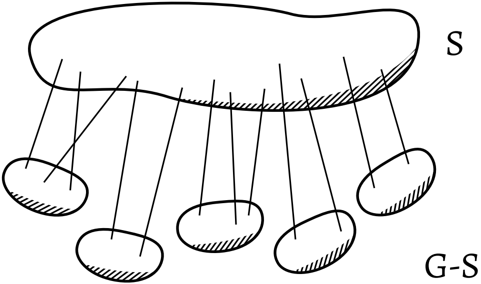
Reduced instance: large protrusions are gone

Small-degree components



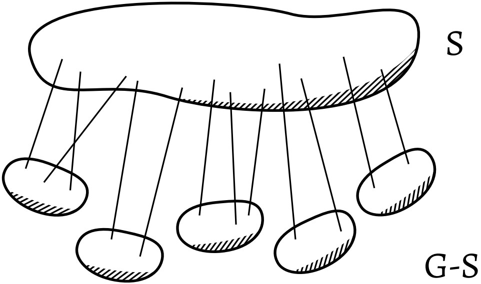
- $D_S(C) < r$, therefore boundary of size r

Small-degree components



- $D_S(C) < r$, therefore boundary of size r
- C has constant treewidth (problem is treewidth bounding)

Small-degree components



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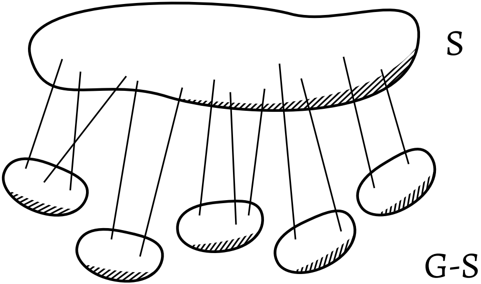
⇒ Each small-degree component has constant size (reduced instance)

Small-degree components

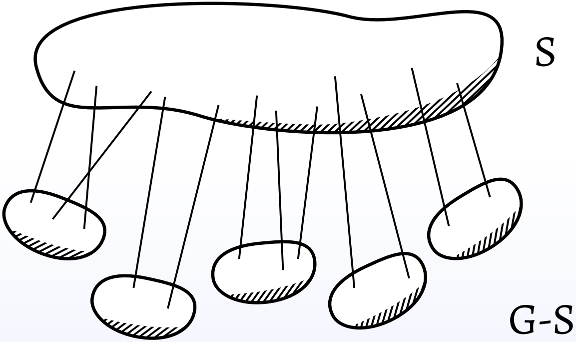


- $D_S(C) < r$, therefore boundary of size r
- C has constant treewidth (problem is treewidth bounding)

$\Rightarrow$ Each small-degree component has constant size (reduced instance)

- What about the *number* of small-degree components?

Small-degree components



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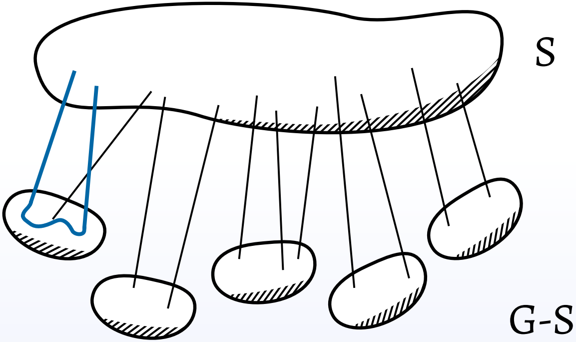

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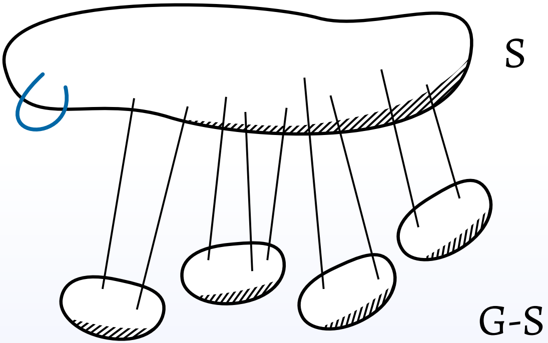

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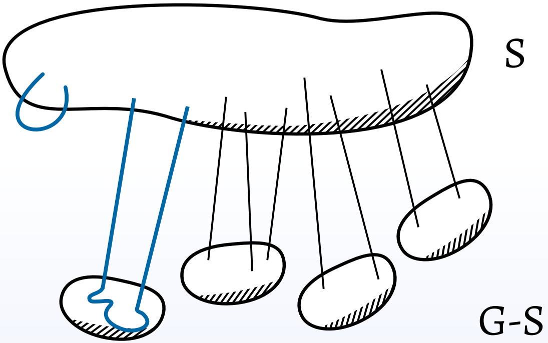

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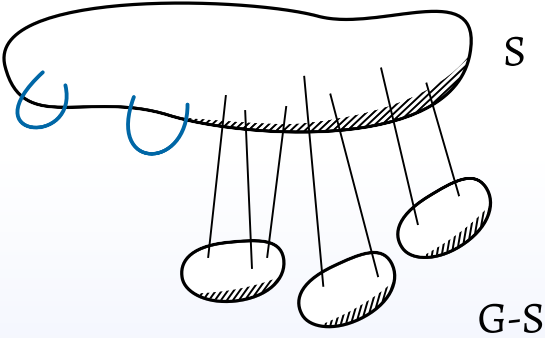

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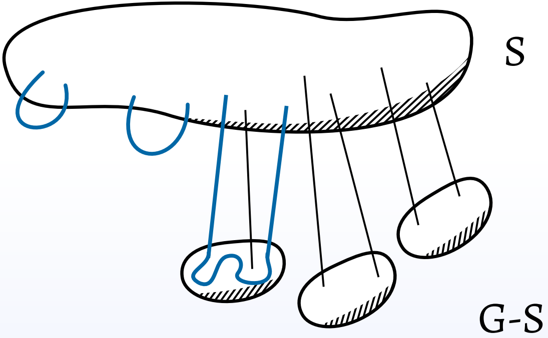

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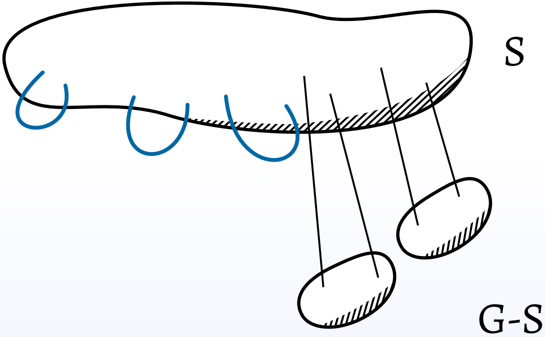

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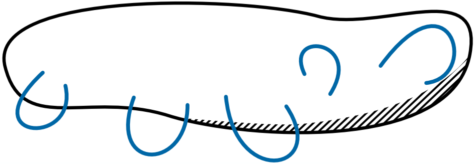

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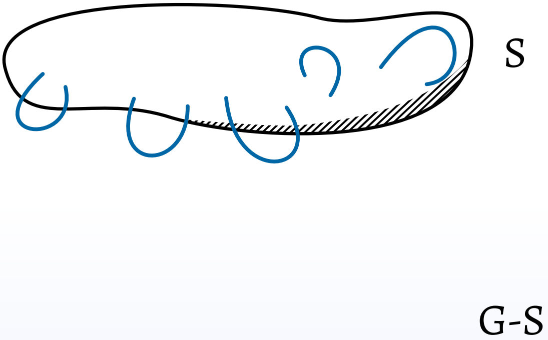
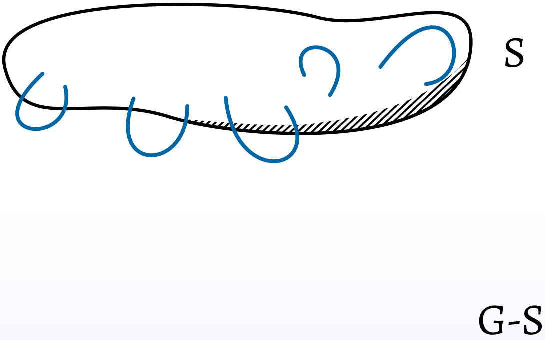


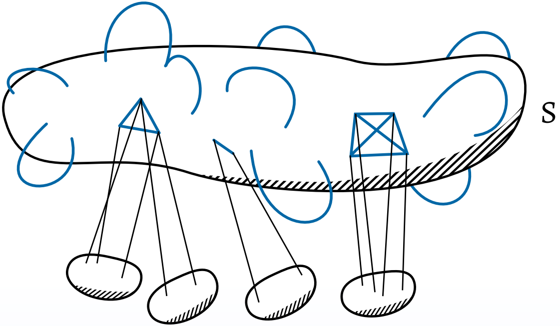
- How often can we do this?

Small-degree components



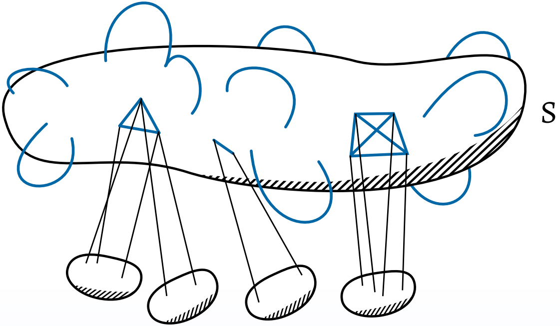
- How often can we do this?
- Is it exhaustive?

Small-degree components



- Components now connected to cliques (or not finished)

Small-degree components



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- $G[S]$ is H-topological minor free, therefore...

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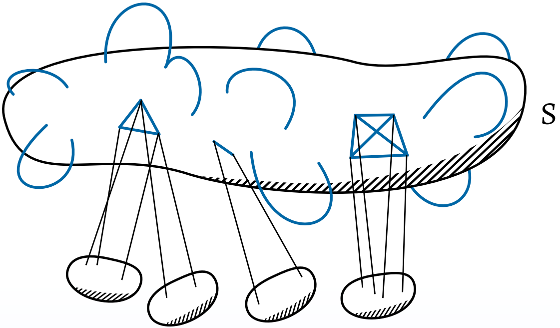

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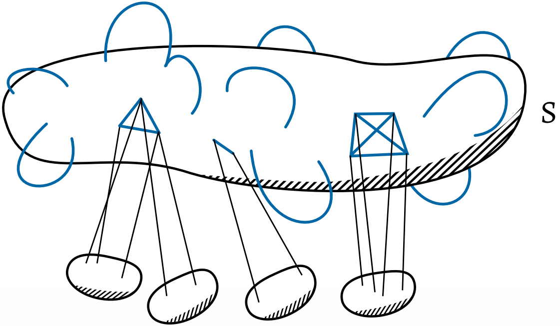

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  ... $O(|S|) = O(k)$ edges

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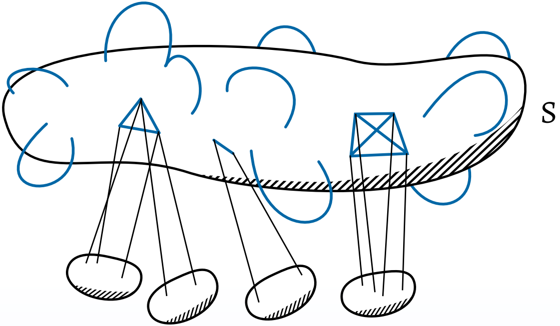

- Components now connected to cliques (or not finished)
- $G[S]$ is H-topological minor free, therefore...
  ... $O(|S|) = O(k)$ cliques
  ... $O(|S|) = O(k)$ edges
- Constant number of vertices in components connected to a common clique (or large protrusion in G)

50% done!

(not)

$O(k)$ vertices in small-degree components

Large-degree components

Very technical. Two ingredients:

Large-degree components

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1. At most $O(k)$ connected *subgraphs* with $D_S \geqslant r$

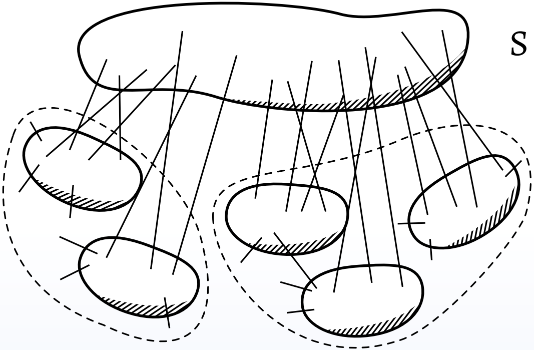
Large-degree components

Very technical. Two ingredients:

1. At most $O(k)$ connected *subgraphs* with $D_S \geqslant r$
2. Tree-decomposition allows us to find many such subgraphs *of constant size*

Large-degree components

Ingredient one



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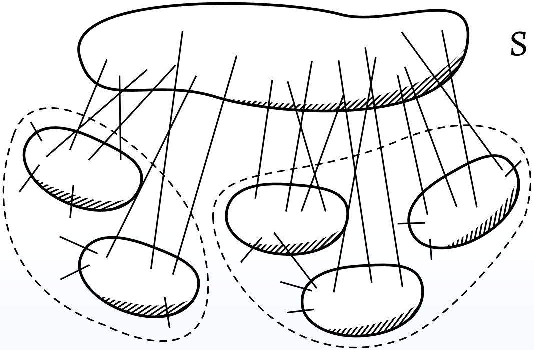

- Same idea as before: contract connected subgraphs into edges in S

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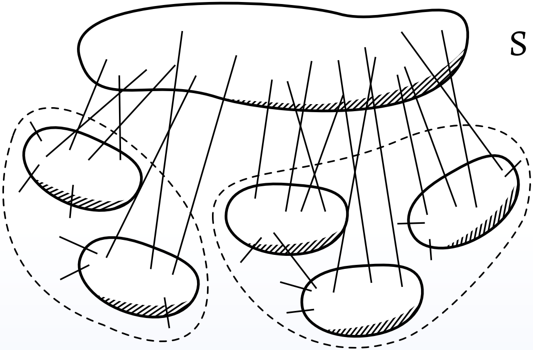
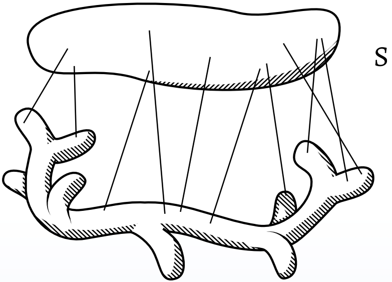


- Same idea as before: contract connected subgraphs into edges in S
- Exhaustive, else K_r as a subgraph in S and thus H as a topological minor in G

Large-degree components

Ingredient two



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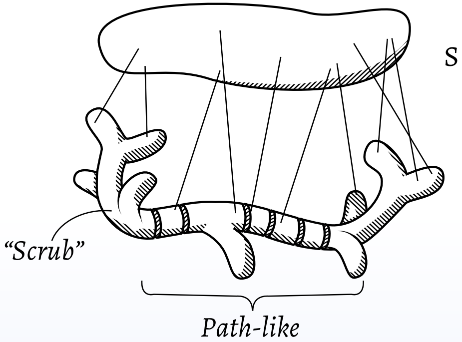

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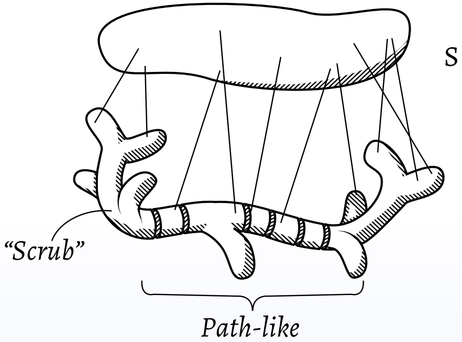
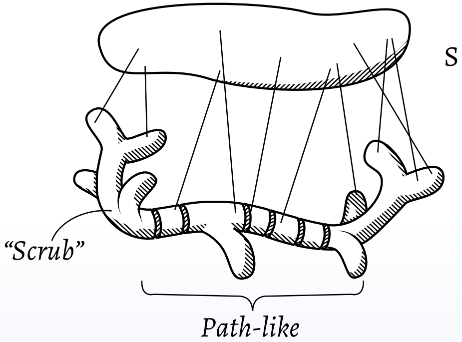


- Walk along path-decomposition

Large-degree components

Ingredient two



- Walk along path-decomposition
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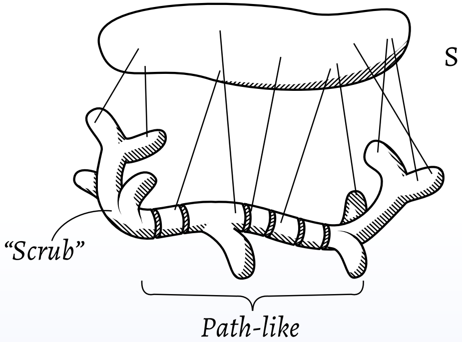

- Walk along path-decomposition
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- If more than $\varpi(2t + r)$ vertices seen: subgraph has large degree wrt S

Large-degree components

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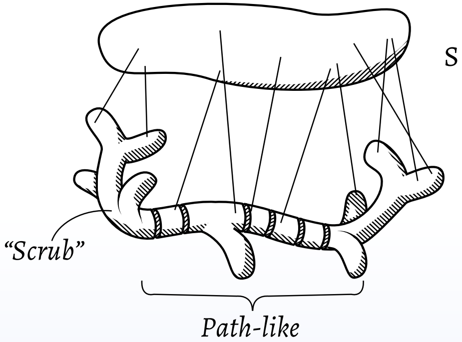

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Conclusion

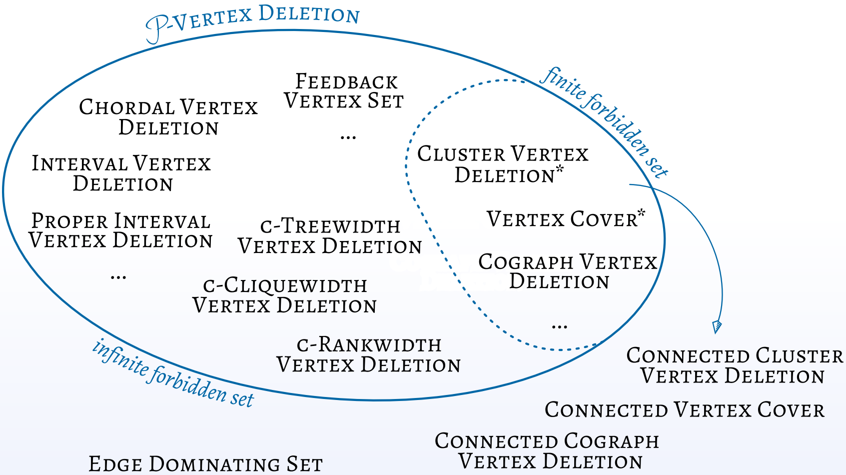
The result

We have shown that problems. . .

1. . . . that can be solved in polynomial time via dynamic programming on graphs of bounded treewidth
2. . . . that have finite integer index
3. . . . and that are treewidth bounding

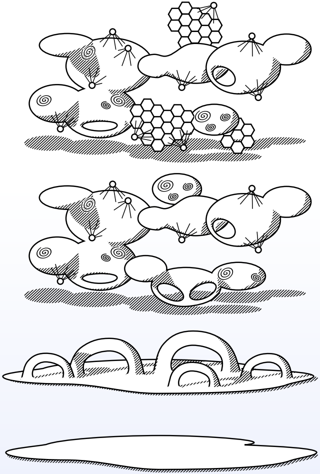
admit linear kernels on graphs excluding a fixed topological minor.

Examples



𝒫-Vertex Deletion
finite forbidden set
infinite forbidden set
Feedback Vertex Set
Chordal Vertex Deletion
...
Interval Vertex Deletion
Proper Interval Vertex Deletion
...
c-Treewidth Vertex Deletion
c-Cliquewidth Vertex Deletion
c-Rankwidth Vertex Deletion
Cluster Vertex Deletion*
Vertex Cover*
Cograph Vertex Deletion
...
Connected Cluster Vertex Deletion
Connected Vertex Cover
Connected Cograph Vertex Deletion
...
Edge Dominating Set

Trade-off: class of instances vs. problem requirements



H-Topological-Minor-Free	*Treewidth-bounding*
∪	
H-Minor-Free	*Bidimensional* *+ separation property*
∪	
Bounded Genus	*Quasi-compact*
∪	
Planar	*“Distance-property”*

Open questions

- What about graphs excluding a fixed *induced minor*/*contraction*/*immersion*? Which other notions of sparse graphs allow such a theorem?
- Can we do this for DOMINATING SET and similar problems? (Grohe & Marx -decomposition!)
- Are there interesting *polynomially* treewidth bounding problems? (We looked at *linear* treewidth bounding)

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Thank you!