Planar Separator Theorem

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Motivation

1. Dividing the problem recursively into smaller ones of the same type can speed up their solving substantially.
2. Some already known examples may be:
   - Merge sort
   - Quicksort
3. Enables for even quicker solution using parallel programmig to solve the subproblems simultaneously.
4. Easier memory access, because a small enough subproblem can be solved only using the cache for example.

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Some useful theorems

Theorem 1

Any closed cureve in the plane devides it into an "inside" and "outside" region

Theorem 2 (Kurtakowski's theorem)

A graph is planar if and only if it contains neither K_5 or $K_{3,3}$ as a generalized subgraph

Theorem 3

Shrinking any edge to a sindle vertex preserves planarity.

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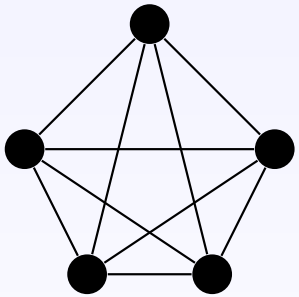

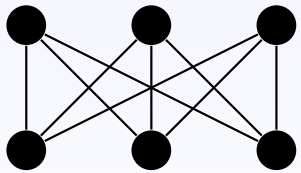

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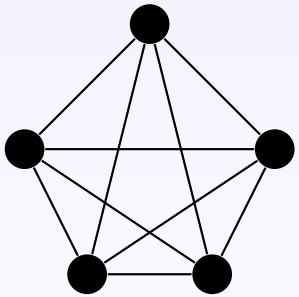

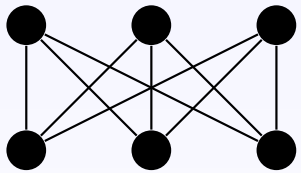

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Given

- G is a connected planar graph
- G's vertices have nonnegative costs summing to no more than one
- G contains a spanning tree with radius r

Result

then we can partition the vertices so that:

- no edges connect A and B
- A and B both cost at most $2/3$
- C contains at most $2r + 1$ vertices

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Proof of Lemma 1

1. Embed G in the plane.
2. Triangulate.
3. Every nontree edge forms a simple cycle with some of the tree edges.
4. This cycle devides the plane into 2 parts. (Theorem 1)
5. At least one of those cycles separates the graph into parts, whose cost does not exceed $2/3$
6. The length of this cycles is at most $2r + 1$ where r is the radius of the spanning three.

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Lemma 2 (Leveling)

Given

- G is a connected planar graph
- G's vertices have nonnegative costs summing to no more then one
- G's vertices are partitioned into levels according to their distance from some vertex v
  - $L(l)$ is the number of vertices on level l
  - r (radius) is the maximum distance from v
  - Let $r+1$ be an additional layer without any vertices
- Given l_1 and l_2 such that
  - Vertices on levels 0 through $l_1 - 1$ have total cost not exceeding 2/3
  - Vertices on levels $l_2 + 1$ through $r + 1$ have total cost not exceeding 2/3

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Then we can partition the vertices so that:

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- C contains at most $L(l_1) + L(l_2) + max\{0, 2(l_2 - l_1 - 1)\}$ vertices

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Proof of Lemma 2

1. If $l_1 \geq l_2$ then the lemma is true
   - A would contain the vertices on levels 0 through $l_1 - 1$ (cost is less than $2/3$)
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2. If $l_1 < l_2$ the lemma still hold true
   - Delete all vertices on levels l_1 and l_2, partitioning the graph into 3 part. (Only the middle one can have a cost exceeding 2/3)
   - If it costs more than 2/3:
     - delete all the vertices on l_2 and above
     - shrink those on l_1 and below to a single one.
   - Then the new graph contains a spanning tree with radius $l_2 - 1 - l_1$ and we can apply lemma 2, which results in A^*, B^*, C^*
   - Then A would be assigned A^* or B^* depending on which has higher cost.
     C gets the vertices on l_1 and l_2 as well as C^*.
     And B is the rest.

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Theorem 4

Given

- G is a n-vertex planar graph
- G's vertices have nonnegative costs summing to no more than one

Result

Then G's vertices can be partiotioned into the sets A, B and C so that:

- A and B both cost at most $2/3$
- C contains no more than $2\sqrt{2}\sqrt{n}$ vertices

Proof of theorem 4

1. If G is connected, the theorem is obviously true, because we can layer the graph and apply lemma 2
2. If G is not connected(Let $G_1, G_2, G_3..., G_k$ with vertices $V_1, V_2, ..., V_k$ respectively). Then we have the following cases:
   - No component has cost exceeding 1/3. Then let i be the minimal index so that $A = V_1 \cup V_2 \cup ... \cup V_i$'s cost exceeds 1/3. Then $B = V_{i+1} \cup V_{i+2} \cup ... \cup V_k$ and $C = \emptyset$
   - If some G_i has cost between 1/3 and 2/3, then we simply make $A = V_i$, B would be left with all the other vertices that arent in A and $C = \emptyset$
   - If some G_i's cost exceeds 2/3 we do step 1, resulting in A^*, B^*, C^*. Then let A be the greater between A^* and B^*, $C = C^*$ and B be the rest.
     ($A = \frac{2}{3} * \frac{2}{3} = \frac{4}{9} < \frac{6}{9} = \frac{2}{3}$ in the worst case and $B = \frac{5}{9} < \frac{6}{9}$ in this case)

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The Algorithm

1. Find a **planar embedding** of G
2. Find the **connected componenents** and determine the cost of each one.
   - If none of the components costs exceed 2/3 we can apply Theorem 4 and be done.
   - Else we have to go to step 3.
3. Do a breadth-first **spanning tree** of the most costly component. Compute the **level of each vertex** and the number of vertices on each level.
4. Find **level l_1** such that
   - the total cost on levels 0 to $l_1 - 1$ is under 1/2
   - the total cost on levels 0 to l_1 is over 1/2
   - let k be the number of vertices on levels 0 to l_1

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5. Find the **highest level** $l_0 \leq l_1$ such that $L(l_0) + 2(l_1 - l_0) \leq 2\sqrt{k}$. Find the **smallest level** $l_2 \geq l_1 + 1$ such that $L(l_2) + 2(l_2 - l_1 - 1) \leq 2\sqrt{n-k}$
6. **Delete** all vertices on levels l_2 and above. **Shrink** all vertices on levels 0 through l_0.
7. Do a breadth-first **spanning tree** rooted at x. Record all the parent of each vertex and the total cost of all descendents + the cost of the vertex itself. **Triangulate**.
8. Let (v_1, w_1) be a nontree edge and find a **cycle** by following parents. Find the cost "inside" and "outside" the cycle by scanning the .
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Maximum independent set (NP-complete)

Theorem 5

Given is:

- Planar n-vertex graph G with nonnegative vertex costs
- $0 \leq \varepsilon \leq 1$

Then we can find:

- A set C of $\mathcal{O}(\sqrt{n/\varepsilon})$ whose removal leaves G with no connected component of cost exceeding ε

C can be found in $\mathcal{O}(n \log n)$

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Proof of Theorem 5

1. Find a separation of G by applying Theorem 4 until there are no more connected components with cost more than ε
2. The runtime is $\mathcal{O}(n \log n)$ as there we have $\mathcal{O}(\log n)$ separations each done in a linear time
3. C is $\mathcal{O}(\sqrt{n/\varepsilon})$ because we have a separator of size $\mathcal{O}(\sqrt{n})$ which has to be calculated for every separation (number of separations $= \mathcal{O}(n/\varepsilon)$)

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So using Theorem 5 for $\varepsilon = \log n/n$ and each vertex having cost $1/n$ the maximum independent set can be computed in $\mathcal{O}(n^2)$-time with an error rate of $\mathcal{O}(1/\sqrt{logn})$

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Embedding data structures

Theorem 6

Any planar graph with maximum degree k can be embedded in a binary tree so that the average proximity is $\mathcal{O}(k)$

Proof:

1. If G has one vertex then T would be a tree of one vertex.
2. Otherwise we apply Theorem 4 with each vertex costing $1/n$, resulting in A, B, C
3. We choose one of the vertices in C and recursively embed the subgraph induced by $(A \cup C) \backslash \{v\}$ in a binary tree T_1
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5. Let T consist of a root (v's image) with two children, the root of T_1 and the root of T_2
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