Parameterized Algorithms WS 2021
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Exercise Sheet with solutions 09

In this exercise sheet we take a look at the *maximum internal spanning tree problem*. The problems asks at most how many internal vertices a spanning tree of a given graph G can contain. We consider the paramterized problem p-IST that is parameterized by the number of internal vertices k.

Task T28

Find a polynomnial time algorithm that finds in a graph G either a spanning tree with k internal vertices or an independent set of size $2n/3$ for $n > 3k$.

Solution

Apply a DFT algorithm to any vertex. If the DFT finds a spanning tree with k internal vertices, we are done, otherwise, the spanning tree must have $n - k + 1$ leaves. Observe, that no two leaves can be adjacent to each other except for the root, by the DFT construction.
Finally, because $n > 3k$, $n - k \geq n - n/3 \geq 2n/3$.

Task T29

Find a kernel of size $3k$ for p-IST. Use the result of Exercise T28 and the following lemma:

Lemma 1. *If $n \geq 3$, and I is an independent set of G of cardinality at least $2n/3$, then there are nonempty subsets $S \subseteq V \setminus I$ and $L \subseteq I$ such that*

1. *$N(L) = S$,*
2. *$B(L, S)$ has a spanning tree such that all vertices fo S and $|S| - 1$ vertices of L are internal.*

Moreover, given a graph on at least 3 vertices and an independent set of cardinality at least $2n/3$, such subsets can be found in time polynomial in the size of G.

The bipartite graph $B(S, L)$ describes the graph induced by G on $S \cup L$ without edges between vertices of S or between vertices of L.

Solution

The kernel uses the given lemma a in a set of reduction rules in the following way.

- **Rule 1:** If the graph has $3k$ vertices or less, it is already a kernel.
- **Rule 2:** Apply DFS. If a spanning tree with k internal vertices is found, we are done, otherwise, we have an independent set of size $2n/3$ as we saw in Exercise T28.
- **Rule 3:** Apply the lemma to the given independent set and find S and L. delete S and L from the graph and subtitute them by a vertex v_S adjacent to the neighbors of S that are not in L, and a vertex v_L adjacent only to v_S. Reduce the parameter by $2|S| - 2$.

Repeat this proceadure until the graph is small enough. For this proceadure to be correct we need to guarantee that the graph gets smaller after applying Rule 3, and that the parameter size does not grow. Let us see that.

Assume that the resulting graph $G_R = (V_R, E_R)$ after applying Rule 3 has a spanning tree with at least $k' = k - 2|S| + 2$ internal vertices if and only if the original graph G has a spanning tree with at least k internal vertices. Indeed, assume G has a spanning tree with $\ell \geq ki$ internal vertices. Then, let $B(S, L)$ be as in the given lemma and T be a spanning tree of G with ℓ internal vertices such that all vertices of S and $|S| - 1$ vertices of L are internal (we do need to prove that such a solution exists, which we do in Exercise H19). Because $T[S \cup L]$ is connected, every two distinct vertices $u, v \in N_T(S) \setminus L$ are in different connected components of $T \setminus (L \cup S)$. But this means that the graph T' obtained from $T \setminus (L \cup S)$ by connecting v_S to all neighbors of S in $T \setminus (S \cup L)$ is also a tree in which the degree of every vertex in $N_G(S) \setminus L$ is unchanged. The graph T'' obtained from T' by adding v_L and connecting v_L to v_S is also a tree. Then, T'' has exactly $\ell - 2|S| + 2$ internal vertices.

In the opposite direction, if G_R has a tree T'' with $\ell - 2|S| + 2$ internal vertices, then all neighbors of v_S in T'' are in different components of $T'' \setminus \{v_S\}$. By the lemma, we know that $B(S, L)$ has a spanning tree T_{SL} such that all the vertices of S and $|S| - 1$ vertices of L are internal. We obtain a spanning tree T of G by considering the forest $T''' = T'' \setminus \{v_S, v_L\} \cup T_{SL}$ and adding edges between different components to make it connected. For each vertex $u \in N_{T''}(v_S) \setminus \{v_L\}$, add an edge uv to T''', where uv is an edge of G and $v \in S$. By construction, we know that such an edge always exists. Moreover, the degrees of the vertices in $N_G(S) \setminus L$ are the same in T as in T''. Thus, T is a spanning tree with ℓ internal vertices. Finally, as $|S| \geq 1$ and $|L \cup S| \geq 3$, we have that $|V_R| < |V|$ and $k' \leq k$.

Task H19 (15pts)

It seems that we overlooked a detail in Exercise T29. To fix it, you have to prove the following lemma:

Lemma 2. *If G has a spanning tree with k internal vertices, then G has a spanning tree with at least k internal vertices which all the vertices of S and exactly $|S| - 1$ vertices of L are internal.*

Is this enough?

Solution

Given a spanning tree T for G with k internal vertices, and given S and L we can build a spanning tree T' with at least k internal vertices containing all of the vertices of S and exacly $|S| - 1$ vertices of L as internal vertices.

First, denote by F the forest obtained from T by removing all edges incident to L. Then, as long as 2 vertices $u, v \in S$ are in the same connected component in F, remove an edge from F that is incident to one of these two vertices and belongs to the $u - v$ path in F. Observe that in F, each vertex from $V \setminus (L \cup S)$ is in the same connected component as some vertex from S. Indeed, we only removed an edge uw incident to a vertex $w \in V \setminus (L \cup S)$ in case $u, v \in S$ and there was a $u - v$ path containing w. After removing uw, w is still in the same connected component as v. Now, obtain the spanning tree T' by adding the edges of a spanning tree of $B(S, L)$ to F in which all vertices of S and $|S| - 1$ vertices of L are internal (see lemma in Exercise T29). Clearly, all vertices of S and $|S| - 1$ vertices of L are internal in T'. It remains to show that T' has at least as many internal vertices as T. Let U be the set of vertices $V \setminus (S \cup L)$, and let i_T denote the number of internal vertices of the tree T. Then, we have that $i_T(L) \leq \sum_{u \in L} d_T(u) - |L|$, as every vertex in a tree has degree at least 1 and internal vertices have degree at least 2. We also have $i_{T'}(U) \geq i_T(U) - (|L| + |S| - 1 - \sum_{u \in L} d_T(u))$ as at most $|S| - 1 - (\sum u \in L d_T(u) - |L|)$

edges incident to S are removed from F to separate $F \setminus L$ into $|S|$ connected components, one for each vertex of S. Thus,

$$\begin{aligned}
i_{T'}(V) &= i_{T'}(U) + i_{T'}(S \cup L) \\
&\geq i_T(U) - (|L| + |S| - 1 - \sum_{u \in L} d_T(u)) + i_{T'}(S \cup L) \\
&= i_T(U) + (\sum_{u \in L} d_T(u) - |L|) - |S| + 1 + i_{T'}(S \cup L) \\
&\geq i_T(U) + i_T(L) - (|S| - 1) + (|S| + |S| - 1) \\
&= i_T(U) + i_T(L) + |S| \\
&\geq i_T(U) + i_T(L) + i_T(S) \\
&= i_T(V) \ .
\end{aligned}$$

Task H20 (5pts)

Use the results above to find a parameterized algorithm that solves p-IST in time $8^k n^{O(1)}$.

Solution

Obtain a $3k$-vertex kernel for the input graph G in polynomial time and run the $2^n \cdot n^{O(1)}$ time algorithm of Nederlof (Fast polynomial-space algorithms using Möbius inversion: Improving onSteiner Tree and related problems. 2009) on the kernel.