RWTH AACHEN UNIVERSITY

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Exercise Sheet with solutions 02

Task T4

Give a polynomial kernel for the following problem.

Input:	A sequence of marbles, each with a non-negative integer weight and color.
Parameter:	A positive integer k.
Question:	Can we remove marbles of total weight at most k, such that for each color, all marbles of that color are consecutive?

Solution

We use the following reduction rules: *Rule 1.* If there are two consecutive marbles of the same color, replace them by a single marble whose weight is the sum of the weights of the two marbles. Call a color *good* if it occurs only once. A marble is good if its color is good. *Rule 2.* If there are two consecutive marbles that are good, replace them with a single marble (with weight that is sum of the weights of the good marbles) and with color of, say, the first of them. Apply both these rules exhaustively. No two consecutive marbles have the same color and no two consecutive marbles are good. Observe that removing a marble can reduce the number of bad colors by at most two: the removed marble itself and the two marbles immediately to the left and right of the removed marble. This gives us *Rule 3:* If there are $2k+1$ bad marbles, say NO. Finally, if a marble has weight at least $k+2$, replace its weight by $k+1$. The good marbles are between bad marbles and hence there are at most $2k+2$ good marbles. Thus the resulting instance has a size of at most $(4k+3)(k+1) = O(k^2)$.

Task T5

We consider the kernel for Vertex Cover by Nemhauser and Trotter. Consider a cycle G with n vertices.

a) How does G_B look like?

b) What are the sets V_0 and C_0?

c) How good is the problem reduction in this case?

Solution

If G is an even cycle, G_B consists of two disjoint cycles of length n each. In the *worst case*, V_0 contains *all* nodes, as a possible optimal vertex cover consists of one side of the bipartition. Hence, the reduction does not make the instance smaller. However, this is only relevant for $n \leq 2k$. For big n, the instance will be rejected by the size guarantee.

If it is odd, G_B is a cycle of length $2n$. Here, for sure an optimal vertex cover is one side of the bipartition.

Task T6

Show that planarity is a hereditary property. Is the forbidden set finite or infinite? If your answer is "finite" then construct the forbidden set; if your answer is "infinite", then construct an infinite family $\mathcal{F}$ of non-planar graphs such that

- for all $G \in \mathcal{F}$, all proper subgraphs of G are planar;
- for all distinct $G_1, G_2 \in \mathcal{F}$, we have that G_1 is not an induced subgraph of G_2.

Solution

To see that there is no finite forbidden set, consider the following family of graphs: take K_5, the complete graph on 5 vertiecs, and replace one edge by a path of lenth l. We denote this graph by K_5^l with $K_5^0 = K_5$.

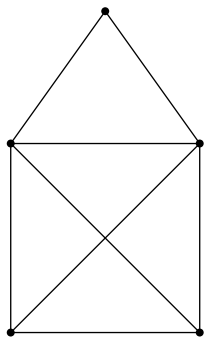
As K_5 is not planar, nor is K_5^l. If there were an embedding of K_5^l in the plane, we could use this embedding for K_5. Furthermore, $K_5^i \not\leq K_5^j$ for $i \neq j$.

Finally, removing a vertex from K_5^l makes it planar: if any of the "original" five vertices is removed, the remaining graph can be embedded like the graph K_4. If one of the "subdivision"-vertices is removed, we use and embedding of the graph without the subdivision vertices and then embed the two pieces of the former path in some face incident to the respective endpoint of the path fragment.

Therefore the family of K_5^l graphs proves that planarity is not characterizable by a finite number of forbidden subgraphs.

Task T7

Does the problem reduction help for the following graph (*Haus des Nikolaus*)? (As in exercise T5)

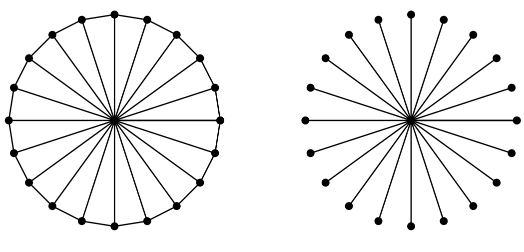


Solution

The graph G_B has a vertex cover number of 5 (this is witnessed by a matching of size 5 with edges $(x, (x+1)')$ and $(5, 1')$). Hence, an optimal vertex cover is one side of the bipartition. The graph is not reduced in this instance.

Task H3 (5 credits)

How does the Nemhauser-Trotter kernel look like for the *wheel* and the *star*?



Solution

The graph G_B of a star is two stars (similar like an even cycle). The optimal vertex cover of it contains both apices. Hence, V_0 is an indepent set (which has a vertex cover number of 0). Here, the reduction works very well.

The case of the wheel is a combination of the star and the cycle. For even n, G_B is two times the n-wheel. In the worst case, V_0 consists of two times C_n. (The apex is contained on both sides, and the other vertices are alternating)

For odd n, G_B is an n-cycle with two additional vertices. The first of those is connected to each even vertex of the cycle, the second one with each odd vertex. The optimal vertex cover consists of one partition. Then V_0 is the whole graph.

Task H4 (5 credits)

Let Π be a decidable parameterized problem. Show that Π is fixed-parameter tractable if and only if there exists a kernelization algorithm for it.

Solution

It is easy to see that a kernel implies a fixed-parameter algorithm, as a brute-force approach on a kernelized instance still runs in time $O(f(k)\, poly(n))$.

For the other direction, assume we have an algorithm that decides Π in time $O(f(k)\, poly(n))$. We will restrict ourselves to computable functions f. The following case distinction will give us a kernel for any instance (I, k):

1. Assume $f(k) < n$. Then the algorithm runs in polynomial time, therefore we can solve (I, k) and output a trivial yes- or no-instance accordingly
2. Assume $f(k) > n$. This already bounds the input size n by some function of k, therefore (I, k) itself is already a problem kernel

Of course this approach usually only gives us exponentially-sized kernels.

Task H5 (5 credits)

Consider the PLANAR INDEPENDENT SET problem: Given a planar graph G and an integer k, decide whether G has an independent set of size at least k. Design an algorithm that takes as input (G, k), where G is planar, and outputs in polynomial time an equivalent instance (G', k') such that $|V(G')| = O(k)$.

Solution

A planar graph is always 4-colorable. This means, there always exists an independent set of size $n/4$. If k is smaller than that, return a trivial yes-instance. Otherwise, the whole graph has size $O(k)$ already, so it is a valid kernel.