Parameterized Algorithms Tutorial

Tutorial Exercise T1

You are given an $n \times n$ matrix M and an integer parameter k. The goal is to select k non-zero entries S such that every other non-zero entry is either in the same row or same column as some element in S. Is this problem in FPT or W-hard? Justify your answer.

Proposed Solution

We reduce to a problem kernel.

Step 1. Delete empty rows and columns and they do not play any role in dominating others.

Step 2. If for a column c, there are at least $k+2$ copies c, keep only $k+1$ of them. Similarly, if for a row r, there are at least $k+2$ copies of r, keep only $k+1$ of them. Since we must select only k non-zero entries in the dominating set, only k columns can actually contribute to the solution and the entries of the remaining columns have to be dominated by non-zero entries in rows. Therefore it suffices to keep only $k+1$ of the columns.

Step 3. If there exist a column that is repeated $k+1$ times and has at least $k+1$ non-zero entries, then the given instance is a NO-instance.

If there is a solution, then the k dominating entries can be rearranged in the upper-left corner of the matrix. The bottom-right corner of the matrix is a block of zeros. Now there can be at most $2^k - 1$ distinct rows, each repeated $k+1$ times and the same for columns. Hence the total size of the matrix is at most $4^k(k+1)^2$.

Tutorial Exercise T2

Consider the following version of the Steiner Tree problem: an input is a graph $G = (V, E)$, a set $S \subseteq V$ and an integer parameter k; the goal is to decide whether there exists a set $T \subseteq V \setminus S$ of size at most k such that $G[T \cup S]$ is connected. Is this problem FPT or W-hard? Justify your answer as usual.

Proposed Solution

The problem is W[2]-complete. For inclusion in W[2], reduce to Short Multitape Turing Machine Acceptance. For W[2]-hardness, reduce from Dominating Set. Given an instance (G, k) of the Dominating Set problem, create a bipartite graph $\tilde{G} = (\tilde{V}, \tilde{E})$ by taking two copies of the vertex set of G, V_1 and V_2, and for each edge $\{u, v\} \in E(G)$, add the edges $\{u_1, v_2\}$ and $\{v_1, u_2\}$, where u_1 and u_2 denote the copies of vertex u in the sets V_1 and V_2. Set $S := V_1$ to be the set of terminals. Then G has a k-dominating set if and only if there exists a set $T \subseteq V_2$ of size at most k such that $\tilde{G}[S \cup T]$ is connected.

Homework H1

You are given a graph G and a positive integer k as parameter. You have to decide whether there exists a vertex set S of size at most k such that $G - S$ is r-regular, where r is a fixed positive integer selected in advance. Is this problem FPT or W-hard? Justify your answer.

Proposed Solution

Delete any vertex of degree at most $r - 1$. Now pick a vertex v of degree at least $r + 1$, select any $r + 1$ of its neighbors, and branch: either v must be deleted from the graph or in any $(r + 1)$-subset of $N(v)$, at least one vertex must be deleted. This yields a simple $O^*((r + 2)^k)$ branching algorithm.

Homework H2

The Dominating Set problem is W[2]-complete in general but in many well-known graph classes it is fixed-parameter tractable. For instance, it has a linear kernel on the class of planar graphs (and, in fact, on graphs of bounded genus, on H-minor-free graphs etc.). A colleague claims that the problem is FPT on bipartite graphs. Would you agree with your colleague? Justify your answer.

Proposed Solution

The Dominating Set problem remains W[2]-complete on bipartite graphs: there is an easy reduction from the same problem in general graphs. Given an instance (G, k) of Dominating Set in general graphs, create a bipartite graph $\tilde{G}$ as follows: create two copies of the vertex set of G and call them V_1 and V_2; let z and z' be two special vertices that are not elements of $V_1 \cup V_2$. The vertex set of $\tilde{G} = (V_1 \cup \{z'\}) \uplus (V_2 \cup \{z\})$ and the edge set of $\tilde{G}$ consists of the following edges:

1. for each $\{u, v\} \in E(G)$, add edges $\{u_1, v_2\}$ and $\{v_1, u_2\}$ in $\tilde{G}$, where u_i, v_i are the copies of u, v in V_i;
2. add edges $\{z, u_1\}$ for all $u_1 \in V_1$;
3. add the edge $\{z, z'\}$.

This completes the edge set construction of $\tilde{G}$.

We claim that G has a dominating set of size k if and only if $\tilde{G}$ has a dominating set of size $k + 1$. For the forward direction, let $S \subseteq V(G)$ be a dominating set of G of size at most k. Let S_1 be the copies of these vertices in V_1. The set $S_1 \cup \{z\}$ dominates all vertices of $\tilde{G}$: vertex z dominates z' and all vertices in V_1; and S_1 dominates all of V_2.

Conversely, if S' is a dominating set of size at most $k + 1$ in $\tilde{G}$, then we may assume without loss of generality that $z \in S'$. For if $z \notin S'$, it must be that $z' \in S'$ and, in this case, we can replace z by z'. Since z dominates all of V_1, we may assume that our dominating set does not contain any vertices from V_2 (because these can only dominate vertices in V_1). Thus the remaining k vertices of S' are from V_1. The originals of these vertices in G then dominate all of $V(G)$.