RWTH Aachen **SS 2014**
Lehrgebiet Theoretische Informatik Problem Set 11
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Parameterized Algorithms Tutorial

Tutorial Exercise T1

Irredundant Set is a W[1]-complete problem. The hardness is difficult to show. In this exercise, we show that it is in W[1]. An instance to this problem is a graph $G = (V, E)$ and an integer parameter k; the question is whether there is a set $V' \subseteq V$ of cardinality k having the property that each vertex $u \in V'$ has a private neighbor. The vertex u has a private neighbor x if $\{u, x\} \in E$ and for all $v \in V' \setminus \{u\}$, $\{v, x\} \notin E$.

Proposed Solution

We reduce to the Short Turing Machine Acceptance Problem. Let (G, k) be an instance of the Irredundant Set problem. Construct a non-deterministic Turing machine T_G whose input alphabet consists of $n + 1$ symbols $\{1, \ldots, n, \#\}$, where $n = |V(G)|$, and whose tape alphabet consists of the blank symbol $\{B\}$ and which works as follows:

1. The machine writes a sequence of $2k + 2$ symbols $a_1, \ldots, a_k, \#, b_1, \ldots, b_k, \#$ on its tape, where $a_i, b_j \in \{1, \ldots, n\}$.
2. It then verifies that the first set of k symbols written are distinct.
3. It then verifies for each j whether b_j is a private neighbor of a_j.

Here is one way to verify whether b_j is a private neighbor of a_j. The machine maintains a counter to count till k and for each $1 \leq j \leq k$, it copies down a_j and b_j after the second # symbol. It first verifies whether $\{a_j, b_j\} \in E$ and then counts the number of neighbors b_j has in the set $\{a_1, \ldots, a_k\}$ (using a second counter). If for some j it finds that b_j has more than one neighbor in $\{a_1, \ldots, a_k\}$, it halts and rejects the input. Otherwise after checking for b_k, it halts and accepts the input.
Steps 1 and 2 take time $O(k)$ and $O(k^2)$. Assuming that the graph G is "hardwired" in the machine as an adjacency matrix, Steps 3 takes time $O(k^2)$. The size of the state space of the machine M and the transition function table can be seen to be polynomial in the size of G. A very rough estimate is as follows: $O(kn)$ states for choosing $2k + 2$ vertices; $O(k)$ states for verifying whether the vertices chosen are all distinct; $O(k)$ states for the counters.

Tutorial Exercise T2

The Induced Matching problem is to decide whether a given graph G has an induced matching of size at least k, where k is the parameter that is supplied as part of the input. While the Maximum Matching problem is polynomial-time solvable, the Induced Matching problem is NP-complete in general. Show by a reduction from Irredundant Set that this problem is also W[1]-hard in bipartite graphs.

Proposed Solution

Let (G, k) be an instance of IRREDUNDANT SET. Construct a graph G' as follows: take two disjoint copies of $V(G)$ and call them V_1 and V_2; for each vertex $u \in V(G)$, add an edge from its copy $u_1 \in V_1$ to its second copy $u_2 \in V_2$; for each edge $\{u, v\} \in E(G)$, add the edges $\{u_1, v_2\}$ and $\{v_1, u_2\}$. This completes the construction of G'. We claim that: G has an irredundant set of size k if and only if G' has an induced matching of size k. Suppose that $S = \{x_1, \ldots, x_k\}$ is an irredundant set in G and let y_i be the private neighbor of x_i. In G', the edges $\{x_{i1}, y_{i2}\}$ form an induced matching. Conversely, let $\{e_1, \ldots, e_k\}$ be an induced matching in G'. Consider two distinct edges $e_i = \{u_{i1}, v_{i2}\}$ and $e_j = \{u_{j1}, v_{j2}\}$. Vertices u_{i1} and v_{j2} are not adjacent and neither are u_{j1} and v_{i2}. Thus if we consider the set $\{u_1, \ldots, u_k\}$, then for each i, v_i is a private neighbor of u_i and hence it is irredundant in G.

Tutorial Exercise T3

We know that CLIQUE is W[1]-complete on general graphs. This does not change when we restrict the graph class to be *regular*. Show that for any fixed integers $\alpha, c \geq 1$, the CLIQUE problem with parameter k remains W[1]-hard on d-regular graphs with $d \geq \alpha k^c$. Conclude that INDEPENDENT SET is W[1]-complete on regular graphs.

Proposed Solution

We reduce from the CLIQUE problem in general graphs. Fix integers α, c. Let (G, k) be an instance of the CLIQUE problem (on general graphs) and let Δ be its maximum degree. Construct a d-regular graph G' with $d \geq \alpha k^c$ as follows:

1. Let $d = \max\{\Delta, \alpha k^c\}$.
2. Take d distinct copies $G_1, \ldots, G_d$ of G, and let v_i denote the vertex in G_i that corresponds to vertex v in G.
3. For every vertex v in G, create a set V_v of $d - \deg(v)$ vertices, and for each $1 \leq i \leq d$, add edges from v_i to each vertex in V_v.

This completes the construction of G'. Note that every vertex in G' has degree exactly d and that G' has at most $2dn$ vertices and $d^2 n$ edges and can be constructed in time $O(d^2 n)$. A subgraph of G' that does not have edges from any G_i is bipartite. Hence G has a k-clique if and only if G' has a k-clique.

Homework H1

Consider the following problem: Given a graph $G = (V, E)$ and integers k and l, decide whether G has k vertices V' such that the cut $(V', V \setminus V')$ has at least l edges. The parameter is k. Show that this problem is W[1]-hard on d-regular graphs, where d is sufficiently large in comparison to k.

Proposed Solution

Reduce from the k-CLIQUE problem on d-regular graphs, where $d > k^2$. A d-regular graph with $d > k^2$ has a k-clique if and only if, it has a vertex cut (V_1, V_2) with $|V_1| = k$ and $|E(V_1, V_2)| = kd - 2\binom{k}{2}$.

Homework H2

Show that Induced Matching remains W[1]-hard on regular graphs. **Hint:** Reduce from the k-Independent Set in regular graphs. Given an instance (G, k) of k-Independent Set where G is regular, construct a graph $\tilde{G}$ as follows: Take two copies of G, say G_1 and G_2, and for each vertex $v \in V(G)$, connect its copy $v_1 \in V(G_1)$ to the closed neighborhood $N[v_2]$ of its copy $v_2 \in V(G_2)$; and, vice versa: connect v_2 to all vertices in $N[v_1]$. Show that G has a k-independent set if and only if $\tilde{G}$ has a k-induced matching.

Proposed Solution