RWTH Aachen **WS 2011/12**
Lehrgebiet Theoretische Informatik Problem Set 5
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Parameterized Algorithms Tutorial

Tutorial Exercise T13

Let G be a graph and let $S \subseteq V(G)$ be some vertex subset. Show that the following properties are MSO-expressible:

- S is a vertex cover of G
- S is an independent set of G
- G is a connected graph
- S induces a cycle in G
- G has a Hamiltonian path
- S induces an even cycle in G

Proposed Solution

- Vertex cover: $vc(S) = \forall x \forall y(\neg xEy \vee x \in S \vee y \in S)$
- Independent set: $is(S) = \forall x \forall y(\neg xEy \vee x \notin S \vee y \notin S)$
- Connected: Let us introduce a slightly more general formula.

  $$con(S) = \forall A \forall B((A \subseteq S \wedge B \subseteq S) \rightarrow \exists a \exists b(a \in A \wedge b \in B \wedge aEb))$$

  Where $con(S)$ means that $G[S]$ is connected ($con(V)$ is the formula we need for this part of the exercise)
- Cycle:

  $$\begin{aligned} cycle(S) = con(S) \wedge \forall x \exists a \exists p \forall y((x \in S \wedge y \in S) \\ \rightarrow a \neq p \wedge a \in S \wedge p \in S \\ \wedge (xEy \rightarrow y = a \vee y = p)) \end{aligned}$$

  If we would leave out the connectedness-condition, our formula would also be satisfied by a collection of disjoint cycles.
- Hamiltonian path:

  $$\begin{aligned} hampath &= \exists F \subseteq E \exists s \exists t\, path(s,t,V,F) \\ path(s,t,S,F) &= \forall x((x \in S \wedge x \neq s \wedge x \neq t) \\ &\quad \rightarrow \exists a \exists p \forall y(a \in S \wedge p \in S \wedge a \neq p \wedge (xFy \rightarrow y = a \vee y = p))) \end{aligned}$$
- Even cycle:

  $$\begin{aligned} evencycle(S) &= cycle(S) \wedge bipartite(S) \\ bipartite(S) &= \exists A \exists B \forall a \forall b((a \in S \wedge b \in S \wedge aEb) \\ &\quad \rightarrow (a \in A \wedge b \in B) \vee (b \in A \wedge a \in B)) \end{aligned}$$

Tutorial Exercise T14

Which of the following graph properties are closed under taking minors?

- Acyclicity
- Chordality
- Planarity
- Bipartiteness
- Connectivity
- bounded degree
- having a vertex cover of size at most k

Proposed Solution

Keep in mind that if a graph class is not closed under taking subgraphs it cannot be closed under taking minors!

- Acyclicity: Closed
- Chordality: Not closed under taking subgraphs
- Planarity: Closed, informally one can keep the embedding of a graph into the plane when contracting an edge
- Bipartiteness: Not closed, we can easily introduce an odd cycle by contracting an edge
- Connectivity: Not closed under taking subgraphs
- bounded degree: Not closed, the maximum degree can easily be increased by edge contractions
- having a vertex cover of size at most k: Closed, can be seen by carefully analysing the contraction operation

Homework H10

Let G be a graph and $S \subseteq V(G)$ some vertex subset. Show that the following properties are MSO-expressible:

- S is a dominating set of G
- S induces a path in G
- S induces an even path in G
- S induces an odd cycle in G
- G is 3-colorable

Proposed Solution
Very similar to the above solution of T13.

Homework H11
Which of the following graph properties are closed under taking minors?

- Bounded diameter
- Bounded average degree
- Distance k to planarity, i.e. one can delete at most k vertices from the graph to make it planar
- 3-Colorability
- excluding some fixed graph H as a minor

The average degree of a graph $G = (V, E)$ is defined as $d_{avg} = \frac{1}{|V|}\sum_{v\in V} d(v)$. A basic result, the handshaking lemma, implies that it can also be calculated as $d_{avg} = \frac{2|E|}{|V|}$

Proposed Solution
In the following, let $G = (V, E)$ be a graph. For $X \subseteq V$, we write $G - X$ for the graph obtained by deleting the vertices of X from G alongside all incident edges.

- Bounded diameter: Not closed under taking subgraphs (we can take disconnected subgraphs)
- Bounded average degree: Not closed under taking subgraphs, imagine a graph of bounded average degree with a large clique as a subgraph.
- Distance k to planarity: As we saw before, planarity is preserved under taking minors. Therefore, assume $X \subseteq V$ is a set of vertices s.t. $G-X$ is planar. If any of the minor-operations is applied to vertices or edges in $G - X$, the resulting minor G' still has the property that $G' - X$ is planar. Now consider minor-operations inside X: deleting a vertex from X, deleting or contracting an edge of G with both endpoints inside X retains the above property (it is simple in all three cases to contstruct a set X' for the resulting minor G' such that $G' - X'$ is planar and $|X'| \leq |X|$).

  Finally, we have to consider operations on edges with one endpoint in X and one in $V \setminus X$. Deleting such an edge again does not destroy the property. What about contraction? Let $uv \in E$ be the edge in question and let $u \in X$ and $v \in V \setminus X$ and let further G_{uv} be the graph obtained by contracting uv in G and let $\bar{v} \in G_{uv}$ be the resulting vertex of the contraction. We argue that $G_{uv} - X'$ is planar, where $X' = (X \setminus \{u\}) \cup \bar{v}$. The reason is simple: $G_{uv} - X'$ is isomorphic to $G - (X \cup \{v\})$, which is a subgraph of the planar graph $G - X$.

  We conclude that "distance to planarity" is indeed a property closed under taking minors.

- 3-Colorability: Not closed. Take any graph that is not 3-colorable, for example K_4. Now subdivide each edge once. The resulting graph is bipartite and therefore even 2-colorable, but it contains a not-3-colorable minor.
- excluding some fixed graph H as a minor: Clearly closed under taking minors. If $G' \preceq G$ contains H as a minor, then so does G.