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WS 2011/12
Problem Set 3
04.11.2011

Parameterized Algorithms Tutorial

Tutorial Exercise T7

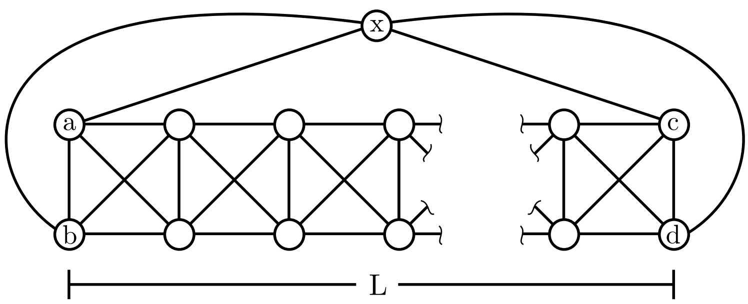
The 4-Coloring problem is the following: given a graph $G = (V, E)$, decide whether there is an assignment of colors to the vertices of G such that adjacent vertices receive distinct colors and the total number of colors used is at most four. This problem is known to be NP-complete.

Show that the property of 4-colorability is hereditary. Does this property admit a finite forbidden set? If not, construct an infinite family $\mathcal{F}$ of non-four-colorable graphs such that

- no two distinct graphs in $\mathcal{F}$ are induced subgraphs of one another;
- for all $G \in \mathcal{F}$, every proper subgraph of G admits a four-coloring.

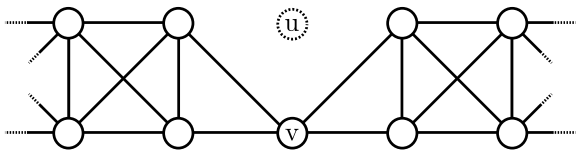
Proposed Solution

Consider the family $\mathcal{F}$ depicted by the following image:



Note that we include only such graphs in $\mathcal{F}$ for which L is even. Clearly, no pair of graphs $G_1, G_2 \in \mathcal{F}$ exists with $G_1 \lneq G_2$. We claim that no $G \in \mathcal{F}$ is four-colorable. Assume a, b are assigned the colors 1 and 2, then every second pair of vertices has these colors, too. It follows that every other pair of vertices (starting with the ones right after a, b) has colors 3 and 4, including the pair c, d. Note that this coloring is unique up to permutations of the colors! ut then x is adjacent to four differently-colored vertices and therefore G cannot be colored with four colors.

What is left to show is that if we remove *any* vertex from a graph $G \in \mathcal{F}$, the remaining graph is four-colorable (this excludes the possiblity that there might be a finite number of forbidden subgraphs which are contained in all the graphs of $\mathcal{F}$). This is obviously true if we remove a,b,c,d or x, so let us look at the other vertices. Say we remove a vertex u. The remaining graph then looks like this:



Assume that u, v had the colors $1, 2$ in a coloring of the original graph, which means that the pairs left and right of u, v, had the colors $3, 4$. Without loss of generatily, let v have the color 1 in that coloring. Then we can color $G - u$ by using the coloring of G, but for all vertices on the right of u, v we exchange color 2 and 3. One can easily verify that this is also a valid coloring. But now, x is adjacent to only three colors, namely $1, 2$ and 4 and we can assign color 3 to it.

It follows that $\mathcal{F}$ has the above properties and therefore the property of being four-colorable does not have a finite forbidden set.

Tutorial Exercise T8

Let k be a constant and consider the class of graphs that have a vertex cover of size *exactly* k. Does this class define a hereditary property? What can you say about the class of graphs that have a vertex cover of size *at most* k? In case you believe that the property is hereditary, how large is the forbidden set?

Proposed Solution

The property asking for a vertex cover of *exactly* size k is not hereditary: especially all graphs of size *smaller* than k do not have this property. As these occur as subgraphs of graphs which have it, the property is note hereditary.

The property asking for a vertex cover of size *at most* k is hereditary by the following argument: say G is a graph with a vertex cover S of size k. Then for every subgraph $G' \leq G$, the set $S \cap V(G')$ is a vertex cover.

We will now prove that a finite set of forbidden subgraph suffices to characterize this property. Consider the set $\mathcal{F}$ of all graphs which have a minumum vertex cover of size $k + 1$. For any such graph G, let S be a vertex cover of that size. As we saw with the Buzz kernel, any vertex of degree at least $k + 2$ must be inside S. We will use this fact to prove that a set of graphs of *bounded size* suffices as a forbidden set, which implies that this set is finite.

First, take all graphs $G \in \mathcal{F}$ where all vertices in S have a degree bounded by $k + 1$. Call this set $\mathcal{F}_{small}$. We can diretly see that this set is finite as it contains graphs of size at most $O(k^2)$.

Now, consider graphs $G \in \mathcal{F}$ where a set of vertices $S' \subseteq S$ have degree larger than $k + 1$. We want to argue that if the degree of a vertex $v \in S'$ is *very* high, we can find a subgraph of G which also has S as a minumum vertex cover—which of course means that G does not need to be included in the forbidden set.

Suppose there exists a vertex $v \in V(G) \setminus S$ which is not adjacent to any vertex of $S \setminus S'$ such that in $G - v$, all vertices of S' *still* have degree $> k + 1$. Then clearly, S is still a vertex cover of $G - v$ and no smaller vertex cover can exist: the vertices of S' must

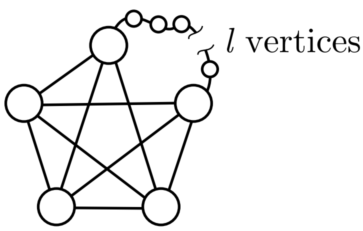
be included in it anyway, as their degree is still large, and the vertices of $S \setminus S'$ are not affected by the removeal of v (any smaller cover of $G - v$ would imply a smaller cover for G which contradicts our initial assumption).
Now, we build our set $\mathcal{F}_{large}$ as follows: include any graph from $\mathcal{F} \setminus \mathcal{F}_{small}$ which does *not* have such a vertex.

Homework H5

Show that planarity is a hereditary property. Is the forbidden set finite or infinite? If your answer is "finite" then construct the forbidden set; if your answer is "infinite", then construct an infinite family $\mathcal{F}$ of non-planar graphs such that

- for all $G \in \mathcal{F}$, all proper subgraphs of G are planar;
- for all distinct $G_1, G_2 \in \mathcal{F}$, we have that G_1 is not an induced subgraph of G_2.

To see that there is no finite forbidden set, consider the following family of graphs: take K_5, the complete graph on 5 vertiecs, and replace one edge by a path of lenth l. We denote this graph by K_5^l with $K_5^0 = K_5$.



As K_5 is not planar, nor is K_5^l. If there were an embedding of K_5^l in the plane, we could use this embedding for K_5. Furthermore, $K_5^i \not\leq K_5^j$ for $i \neq j$.
Finally, removing a vertex from K_5^l makes it planar: if any of the "original" five vertices is removed, the remaining graph can be embedded like the graph K_4. If one of the "subdivision"-vertices is removed, we use and embedding of the graph without the subdivision vertices and then embed the two pieces of the former path in some face incident to the respective endpoint of the path fragment.
Therefore the family of K_5^l graphs proves that planarity is not characterizable by a finite number of forbidden subgraphs.

Homework H6

A graph $G = (V, E)$ is a *divorce graph* if its vertex set can be partitioned into sets $X \uplus Y$ such that $G[X]$ is a complete graph and $G[Y]$ has no edges (there can be edges between the sets X and Y). Show that a graph is a divorce graph if and only if it does not contain the following two graphs as induced subgraphs: