Tutorial Exact Algorithms

Exercise T25 An $n \times n$ matrix is *stochastic* if all its entries are nonnegative and every row sums up to 1. An $n \times n$ matrix is *doubly stochastic* if its entries are nonnegative and each row and each column sums up to 1.

1. Let M be the incidence matrix of a regular bipartite graph of degree $r \geq 1$. Show that $\frac{1}{r}M$ is doubly stochastic.
2. The Permanent Inequality states that for any doubly stochastic matrix A

$$\operatorname{perm}(A) \geq \operatorname{perm}\left(\frac{1}{n}J_n\right) \geq \frac{n!}{n^n},$$

where J_n is the all-ones $n \times n$ matrix. This inequality was conjectured by van der Waerden in 1926 and proved independently by Egorichev and Falikman in 1980. Use this inequality to show that a regular bipartite graph of degree r has at least $(r/e)^n$ perfect matchings.

Exercise T26 The Hamiltonian Cycle Problem. Given a graph $G = (V, E)$ decide whether G has a simple cycle $(u_1, u_2, \ldots, u_{|V|}, u_1)$ containing all vertices. Formulate this problem as an Inclusion-Exclusion formula and show that it can be solved in time $O^*(2^n)$ and polynomial space.

Homework Assignment H26 (10 Points) Given an $n \times n$ matrix $A = [a_{ij}]$ show that

$$\operatorname{perm}(A) = \sum_{S \subseteq [n]} (-1)^{|S|} \prod_{i=1}^{n} \sum_{j \notin S} A[i, j].$$

Hint. One can view the elements of A as variables and the permanent of A as a multinomial polynomial. One way of proving the above is by determining the coefficient of these terms on either side of the purported identity. For example, for a fixed permutation $\pi\colon [n] \to [n]$, the coefficient of the term $a_{1,\pi(1)}, \ldots, a_{n,\pi(n)}$ is a 1 on either side. Now determine the coefficient of terms that involve variables from k columns. The coefficient of such a term is 0 on the left-hand side. What is the coefficient on the right-hand side? Conclude that the permanent of an $n \times n$ matrix can be computed in time $O^*(2^n)$ and space polynomial in n.

Homework Assignment H27 (10 Points) The Travelling Salesperson Problem (TSP). Given a graph $G = (V, E)$ where $V = \{s, u_1, \ldots, u_n\}$ and a weight function $w\colon E \to \{1, \ldots, c\}$, determine the weight of the smallest tour that starts and ends at s. Use Inclusion-Exclusion to solve TSP in time $O^*(2^n c)$. What is the space requirement of your algorithm? Comment on whether it is a polynomial space algorithm.