

Tutorial Exact Algorithms

The hints for the exercises and assignments are to found in the slides for the lectures.

Exercise T29 PROPERTIES AND USES OF THE CONVOLUTION PRODUCT. Let U be a set of n elements and let f, g, h be functions from 2^U to \mathbf{Z} . Recall that the convolution of f and g , denoted by $f * g$, is a function from 2^U to \mathbf{Z} defined as follows. For all $S \subseteq U$,

$$(f * g)(S) = \sum_{\substack{X, Y \subseteq S \\ X \cap Y = \emptyset}} f(X)g(Y).$$

Show that the convolution product is commutative and associative, that is,

$$\begin{aligned} f * g &= g * f. \\ (f * g) * h &= f * (g * h). \end{aligned}$$

Given a graph $G = (V, E)$, define $f: 2^V \rightarrow \{0, 1\}$ as follows. For $S \subseteq V$, we let $f(S) = 1$ iff S is an independent set in G .

1. Show that $(f * f)(V)$ represents the number of two colorings of the graph G .
2. What does the iterated k -fold convolution $(f * \dots * f)(V)$ (k times) represent? How would you compute the chromatic number of a graph G , if you were given an algorithm that computes the convolution $f * g$?
3. If $f(S)$ is defined to be 1 iff S is a dominating set of G , then what does the iterated k -fold convolution $(f * \dots * f)(V)$ (k times) represent?

Exercise T30 FAST RANKED ZETA TRANSFORM. In class we saw how to compute the zeta and Möbius transforms of a function $f: 2^U \rightarrow \mathbf{Z}$ in time $O^*(2^{|U|} \cdot \log m)$ where m is a positive integer such that for all $S \subseteq U$, it holds that $|f(S)| \leq m$. The objective of this exercise is to show that the ranked version of the zeta transform can also be computed within this time. As usual, we assume that $f(S)$ for all $S \subseteq U = [n]$ can be computed in time $O^*(2^n \cdot \log m)$ using space $O^*(2^n \cdot \log m)$. The steps of the fast algorithm for the ranked zeta transform are as follows:

1. First compute the values of $f(S)$ for all $S \subseteq U$ and store them in a table of size $O^*(2^n \cdot \log m)$.
2. For $0 \leq i \leq n$, define $\zeta_k^{(i)}(f)$ as follows: for all $S \subseteq [n]$,

$$\zeta_k^{(i)}(f)(S) = \sum_{\substack{X \subseteq S; |X|=k \\ S \setminus [i] \subseteq X}} f(X).$$

What do $\zeta_k^{(0)}(f)(S)$ and $\zeta_k^{(n)}(f)(S)$ represent?

3. If $i \in S$, show that

$$\zeta_k^{(i)}(f)(S) = \zeta_k^{(i-1)}(f)(S \setminus i) + \zeta_k^{(i-1)}(f)(S).$$

4. Now write down a recurrence for $\zeta_k^{(i)}(f)(S)$ in terms of $\zeta_k^{(i-1)}(f)(S \setminus i)$ and $\zeta_k^{(i-1)}(f)(S)$.

5. What is the time and space taken to compute $\zeta_k^{(i)}(f)(S)$ for all i, k , and S ?

Homework Assignment H30 (10 Points) COUNTING k -PARTITIONS. Let U be a set of n elements and let \mathcal{F} be a family of subsets of U . A k -partition of the set system (U, \mathcal{F}) is a collection S_1, \dots, S_k of k sets from the family \mathcal{F} such that $S_i \cap S_j = \emptyset$ for all $i \neq j$ and $\bigcup_{i=1}^k S_i = U$. Show that the number of k -partitions of a set system can be written as the iterated k -fold convolution of an appropriate function. This will show that an algorithm for computing the convolution of two functions may be used to count the number of k -partitions of a set system.

Homework Assignment H31 (10 Points) THE MÖBIUS AND ZETA TRANSFORM. In class we saw that the Möbius transform of the zeta transform of a function $f: 2^U \rightarrow \mathbf{Z}$ is the function f itself. That is, for all $S \subseteq U$

$$\mu(\zeta(f))(S) = \sum_{X \subseteq S} (-1)^{|S \setminus X|} \zeta(f)(X) = f(S).$$

Show that the converse also holds, that is, the zeta transform of the Möbius transform of f is the function f itself.

Solution To prove that $\zeta(\mu(f)) = f$, it is sufficient to show that

$$f(S) = \sum_{X \subseteq S} \mu(f)(X) \tag{1}$$

for all $S \subseteq U$. By definition of the Möbius transform, the rhs of (1) may be written as

$$\begin{aligned} \sum_{X \subseteq S} \mu(f)(X) &= \sum_{X \subseteq S} \sum_{Y \subseteq X} (-1)^{|X \setminus Y|} f(Y) \\ &= \sum_{Y \subseteq S} \left(\sum_{\substack{X \subseteq S \\ Y \subseteq X}} (-1)^{|X \setminus Y|} \right) f(Y). \end{aligned}$$

The outer sum is over all subsets Y of S while the inner sum is over all supersets X of Y . Let $|Y| = p$ and $|S| = k$. If $k > p$ then the inner sum evaluates to

$$\sum_{i=0}^{k-p} (-1)^i \binom{k-p}{i} = (1-1)^{k-p} = 0.$$

If $p = k$ then the inner sum evaluates to 1 and the rhs of (1) is $f(S)$.

Homework Assignment H32 (10 Bonus Points) FAST RANKED MÖBIUS TRANSFORM. Given a function $f: 2^{[n]} \rightarrow \mathbf{Z}$, show that the ranked Möbius transform of f can be computed in time $O^*(2^n \cdot \log m)$ time, where $m \in \mathbf{N}$ is such that for all $S \subseteq [n]$, it holds that $|f(S)| \leq m$. As usual, assume that for all $S \subseteq [n]$ one can compute $f(S)$ in time $O^*(2^n \cdot \log m)$ and space $O^*(2^n \cdot \log m)$.