Analysis of Algorithms, WS 2020
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Exercise Sheet with solutions 08

The Gardener's Tale:

Just a christmas tradition
a gift exchange of sorts
a present for a person
and one rule to enforce

Each of us buys a present
only one and no more
who is your secret santa?
yourself? that'd be a bore!

We have to write our names
on a little piece of paper
one at random you may choose
what have we got to loose?

Out of n persons and with great care
how many would we expect to despair?

Having chosen themselves
as secret santa instead
their own name they read
and restart again with dread

Out of n persons and with great caution
Many times we give the names,
yes, but how often?

The Politician's Tale:

If you want to liven up your home
With lovely decoration
You may want to set a tone
To raise your loved ones expectations

Use christmas ornaments and balls
But beware of superstition
'Cause even the best decorator falls
If others detect repetition

If you are color blind
And still want to decorate
Be sure not to find
Any ornaments to replicate

Old christmases display
Not any of the n
For there is dismay
For more than a span

The Physicist's Tale:

There once was a man in postal office
Who always read a book while he drank his coffee
But one day the novel he liked went missing
So instead he read letters about lovers kissing
On the first letter Jane wrote to Austin
Who was in a lecture on the Higgs boson
The second letter opened led to a third
And be before he knew it n letters were read
The letters and the envelopes were all so scattered
The empty cup of coffee seemed to hardly matter
Now opening letters could send him to prison
He had to find a way to find rhyme and reason
He figured he could choose a letter at random
And prayed that the cover was the right one
So what are the chances, what are the odds
That no single letter went to the right spot?

The Bicycle Repairman's Tale:

Long days — Stifling air
Mediocre food — Drinking Mate
Full of ideas — Familiar faces
Have become friends — Brothers in arms
Against weekly problems — The world can go on
When we'll be finally together
At the computer science center

The Policeman's Tale:

Ashore n sailors slunk,
and got awfully drunk.
When they returned,
they were unconcerned,
fell asleep in a random bunk.

The Lockpick's Tale:

n criminals want to fix
their finances by using a mix
their crypto to blend
for them then to spend
and not to get caught with their tricks

The Thief's Tale:

It needs to be fixed
Where its place is fine?
This poem will be mixed
Is there a line

Problem T18

A permutation with no fixpoints is called a *derangement*. What is the probability that a random permutation of n items is a derangement? If a program generates random permutations, how long do you have to wait on average until you get a derangement? Before attacking this problem mathematically try your best judgement: Is the probability high or low?

Solution

Let the n gentlemen be labeled $1, 2, \ldots, n$. A permutation of $\{1, \ldots, n\}$ in which element i is not placed at position i, for any i, is called a *derangement*. For example, for $n = 3$, 312 is a derangement but 321 is not as 2 is in the second place.

Let D_n denote the number of derangements of n elements. Clearly $D_1 = 0$. $D_2 = 1$ as 21 is the only derangement. We will define $D_0 = 1$. It is convenient to say that there is one permutation of the empty set and that it does not map anything to itself.

Consider the general case with $n+1$ elements. Element 1 has to be at some position k, where $2 \leq k \leq n+1$. Now there are two possibilities. Either element k is at position 1, in which case there are D_{n-1} derangements possible. Otherwise, some other element is at position 1. This second situation may also be viewed as follows: We keep element 1 fixed at the first position; derange elements $2, \ldots, n+1$ in D_n ways; finally, exchange the elements at the first and kth positions to obtain a derangement of the elements $1, \ldots, n+1$. The recurrence for D_{n+1} may now be written as:

$$D_{n+1} = n(D_n + D_{n-1}). \tag{1}$$

Using the above recurrence, we can write $D_{n+1} - (n+1)D_n$ as:

$$\begin{aligned}
D_{n+1} - (n+1)D_n &= nD_{n-1} - D_n \\
&= -\left(D_n - nD_{n-1}\right) \\
&= (-1)^2 \left(D_{n-1} - (n-1)D_{n-2}\right) \\
&= (-1)^3 \left(D_{n-2} - (n-2)D_{n-3}\right) \\
&\;\vdots \\
&= (-1)^{n-1}(D_2 - 2D_1).
\end{aligned}$$

Put differently, the recurrence (1) may be expressed as:

$$D_{n+1} = (n+1)D_n + (-1)^{n+1} \quad \text{where} \quad n \geq 2. \tag{2}$$

Define $D(z) = \sum_{n=0}^{\infty} D_n \frac{z^n}{n!}$. Multiply both sides by $z^{n+1}/(n+1)!$ and sum over n, obtaining:

$$\sum_{n=0}^{\infty} D_{n+1} \frac{z^{n+1}}{(n+1)!} = \sum_{n=0}^{\infty} (n+1)D_n \frac{z^{n+1}}{(n+1)!} + \sum_{n=0}^{\infty} (-1)^{n+1} \frac{z^{n+1}}{(n+1)!}$$

The left-hand-side is $D(z) - D_0$. The first term on the right-hand-side is $zD(z)$ and the second term is $e^{-z} - 1$. Thus the above equation may be written as:

$$D(z) = \frac{e^{-z}}{1-z} = \sum_{n=0}^{\infty} (-1)^n \frac{z^n}{n!} \sum_{n=0}^{\infty} z^n,$$

from which we may write down D_n as:

$$D_n = n! \left(1 - \frac{1}{1!} + \frac{1}{2!} - \frac{1}{3!} + \cdots + (-1)^n \frac{1}{n!}\right).$$

Thus the probability that no gentleman recieves his own hat is $D_n/n!$ which approaches $e^{-1} = 0.3678\ldots$ This is independent of n.

Problem T19

Find a bivariate generating function and a closed-form expression for the number of bitstrings of length n that contain exactly m ones and do not contain the substring 11.

Solution

Let $b_{n,m}$ be the number of bitstrings that do not contain 11, have length n, and contain m ones. Such bitstrings can be described by the recursive definition $F = 0F + 10F + \varepsilon + 1$.

We get the following generating function:

$$\begin{aligned}
F(u,z) = &\sum_{n,m\geq 0} b_{n,m} u^m z^n \\
=& zF(z) + uz^2 F(z) + 1 + uz \\
=& \frac{1+uz}{1-z-uz^2} \\
=& (1+uz)\sum_{k\geq 0} z^k (1+uz)^k \\
=& (1+uz)\sum_{k\geq 0} z^k \sum_{i=0}^{k} \binom{k}{i} u^i z^i \\
=& (1+uz)\sum_{k\geq 0} \sum_{i=0}^{k} \binom{k}{i} u^i z^{i+k} \\
=& \sum_{k\geq 0} \sum_{i=0}^{k} \binom{k}{i} u^i z^{i+k} + \sum_{k\geq 0} \sum_{i=0}^{k} \binom{k}{i} u^{i+1} z^{i+k+1}
\end{aligned}$$

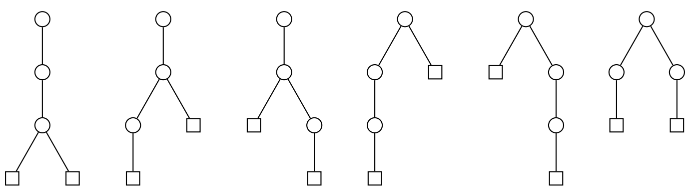
We are interested in the coefficients corresponding to $z^n u^m$. Thus we need to consider the summand with $i = m \wedge i + k = n$ (front) and $i + 1 = m \wedge i + k + 1 = n$ (back). This gives the closed formula for the number of valid bitstrings:

$$\binom{n-m}{m} + \binom{n-m}{m-1} = \binom{n-m+1}{m}$$

Problem H18 (10 credits)

Find a bivariate generating function for the number of (oriented) trees with exactly n internal and m external vertices $T_{n,m}$. For what values of n, m do we have $T_{n,m} = T_{m,n}$?

Example: $b_{3,2} = 6$ and these are the six trees with 3 internal and 2 external nodes:

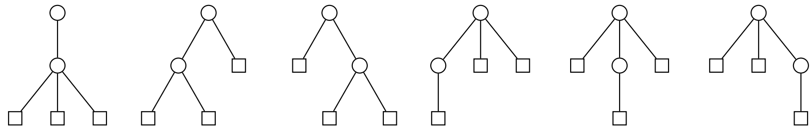


As a warmup exercise try to find and draw all trees with 2 internal and 3 external nodes.

Hint: Do not do all the computations by hand. Seek the help of a computer algebra system. `maxima` or `WolframAlpha` can solve quadratic equations and can find the coefficients of a generating function via Taylor expansion.

Solution

We can draw first all trees with 2 internal and 3 external nodes as a warmup.



And we see that $b_{2,3} = b_{3,2} = 6$.

We consider a recursive definition of trees. A tree is either an external node or an internal node with at least one subtree. This yields

$$T = \square \cup \bigcirc \times T \times T^*$$

und and the generating function

$$T(u,z) = u + z\frac{T(u,z)}{1 - T(u,z)},$$

where u is the number of external and z is the number of internal nodes. We need to solve this equation for $T(u,z)$. We delegate this task to `maxima`: The call `solve(T=u+z*T/(1-T),T)` yields

$$-\frac{\sqrt{z^2 + (-2*u-2)*z + u^2 - 2*u + 1} + z - u - 1}{2}$$

and

$$+\frac{\sqrt{z^2 + (-2*u-2)*z + u^2 - 2*u + 1} - z + u + 1}{2}.$$

We know that for $u = z = 0$ the solution needs to be 0. Thus only the first solution is correct. We get the coefficients by doing a Taylor expansion of $T(u,z)$. We enter

```
T:-((sqrt(z^2+(-2*u-2)*z+u^2-2*u+1)+z-u-1)/2);
taylor(T,[z,u],0,5);
```

into `maxima`. We read the coefficients:

$n+m$	Term
1	u
2	uz
3	$uz^2 + u^2z$
4	$uz^3 + 3u^2z^2 + u^3z$
5	$uz^4 + 6u^2z^3 + 6u^3z^2 + u^4z$

We also want to find out for which values holds $T(u,z) = T(z,u)$. We do so by calculating $T(u,z) - T(z,u)$. We type into `maxima`:

```
TT(u,z):=-((sqrt(z^2+(-2*u-2)*z+u^2-2*u+1)+z-u-1)/2);
ev(TT(u,z)-TT(z,u),expand);
```

The answer is $u - z$. Therefore, the generating function of the difference is $u - z$. The all coefficients of this function are zero except for the case $u = 1, z = 0$ or $u = 0, z = 1$. This means $T_{u,z} = T_{z,u}$ for all other values. This makes sense as there is exactly one tree with a single external node and zero trees with a single internal node.

Problem H19 (10 credits)

Use the symbolic method to calculate the number of words of length n that can be created by the following grammar. Emojis are the terminal symbols and capital letters are variables.

$P \to$ 😀 P 😍 | 🤡 P 🙀 | 💩 | 💩 P

Solution
The language of this grammar can be described by the following recursive definition.

$$P = (\{😀\} \times P \times \{😍\}) \cup (\{🤡\} \times P \times \{🙀\}) \cup \{💩\} \cup (\{💩\} \times P)$$

We need to define the weight of the atomic elements.

$$|😀| = |😍| = |🤡| = |🙀| = |💩| = 1$$

Let $T(z)$ be the generating function for the number of words of length n generated by the grammar. The symbolic method yields

$$T(z) = 2z^2 T(z) + z + zT(z).$$

We transform this into

$$T(z) = \frac{z}{1 - z - 2z^2}.$$

Notice that $1 - z - 2z^2 = (z+1)(1-2z)$. We want to find a partial fraction decomposition of the form

$$\frac{z}{1 - z - 2z^2} = \frac{A}{z+1} + \frac{B}{1-2z}.$$

By setting $z = 0$ we get $0 = A + B$ and by setting $z = 1$ we get $-1/2 = A/2 - B$. Together they yield $A = -1/3$ and $B = 1/3$. This means

$$T(z) = -\frac{1}{3}\frac{1}{z+1} + \frac{1}{3}\frac{1}{1-2z}$$

and therefore $[z^n]T(z) = \frac{1}{3}(2^n - (-1)^n)$. There are $\frac{1}{3}(2^n - (-1)^n)$ words of length n.

Problem H20 (10 credits)
Use the symbolic method to calculate the number of words of length n that can be created by the following grammar with the starting Symbol P.

$$\begin{aligned} Q &\rightarrow 😀 P 😍 \\ P &\rightarrow PQ \mid \varepsilon \end{aligned}$$

Hint: Use the sequence operator.

Solution
The language of this grammar can be described by the recursive definition

$$Q = (\{😀\} \times \bigcup_{i=0}^{\infty} Q^i \times \{😍\}).$$

The symbolic method yields

$$Q(z) = \frac{z^2}{1 - Q(z)}.$$

We solve the quadratic equation

$$Q(z)^2 - Q(z) + z^2 = 0.$$

Only the positive solution is feasible. Therefore

$$Q(z) = \frac{1}{2} + \frac{\sqrt{1-4z^2}}{2} = \frac{1}{2} + \frac{1}{2}(1-4z^2)^{\frac{1}{2}} = \frac{1}{2} + \frac{1}{2}\sum_n \binom{1/2}{n}(-1)^n 4^n z^{2n}.$$

This means the number of words of odd length is 0 and the number of words of length $2n$ is $\frac{1}{2}\binom{1/2}{n}(-1)^n 4^n + \frac{1}{2}(n = 0)$.