Analysis of Algorithms, WS 2020
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Exercise Sheet with solutions 07

Problem T16

This tutorial is geared towards solving recurrence relations using generating functions. Here is an observation: If a_n satisfies the recurrence

$$a_n = x_1 a_{n-1} + x_2 a_{n-2} + \cdots + x_t a_{n-t}$$

for $n \geq t$, then the generating function $a(z) = \sum_{n\geq 0} a_n z^n$ is a rational function $a(z) = f(z)/g(z)$, where the denominator polynomial is $g(z) = 1 - x_1 z - x_2 z^2 - \cdots - x_t z^t$ and the numerator polynomial is determined by the intial values $a_0, a_1, \ldots, a_{t-1}$ and has degree at most $t-1$. In fact, $f(z)$ may be written as:

$$f(z) = g(z) \sum_{n=0}^{t-1} a_n z^n \left(\bmod\ z^t\right).$$

Solve the following recurrence relations:

1. $a_n = 2a_{n-1} + a_{n-2} - 2a_{n-3}$ for $n > 2$ with $a_0 = 0$ and $a_1 = a_2 = 1$.
2. $a_n = 5a_{n-1} - 8a_{n-2} + 4a_{n-3}$ for $n > 2$ with $a_0 = 0$, $a_1 = 1$, and $a_2 = 4$.

Solution

1. First write down $g(z)$ as $1 - 2z - z^2 + 2z^3$. Using the initial conditions, $f(z)$ may be written as:

$$\begin{aligned} f(z) &= g(z) \sum_{n=0}^{2} a_n z^n \ (\bmod\ z^3) \\ &= (z + z^2)(1 - 2z - z^2 + 2z^3)\ (\bmod\ z^3) \\ &= z - z^2 = z(1-z). \end{aligned}$$

Therefore,

$$a(z) = \frac{z(1-z)}{1 - 2z - z^2 + 2z^3} = \frac{z(1-z)}{(1-z)(1+z)(1-2z)} = \frac{z}{(1+z)(1-2z)}.$$

Using partial fractions, the last expression on the right hand side works out to:

$$\frac{z}{(1+z)(1-2z)} = \frac{1}{3}\left(\frac{1}{1-2z} - \frac{1}{1+z}\right),$$

from which we can read off $a_n = \frac{1}{3}(2^n - (-1)^n)$.

2. We start by writing down $g(z)$ as $1 - 5z + 8z^2 - 4z^3$. Now using the initial conditions, $f(z)$ can be written as:

$$\begin{aligned} f(z) &= g(z) \sum_{n=0}^{2} a_n z^n = g(z)(z + 4z^2) \pmod{z^3} \\ &= (1 - 5z + 8z^2 + 4z^3)(z + 4z^2) \pmod{z^3} \\ &= z(1 - z). \end{aligned}$$

Now $1 - 5z + 8z^2 - 4z^3$ can be factorized as $(1 - z)(1 - 2z)^2$, so that $a(z) = z/(1 - 2z)^2$. We know that

$$\frac{2z}{(1 - 2z)^2} = \sum_{n=0}^{\infty} n(2z)^n,$$

and so $a_n = n2^{n-1}$ for $n \geq 3$. Actually, substituting $n = 0, 1, 2$, we see that this holds for $n \geq 0$.

Problem T17

In this problem, we will solve the *Quicksort* recurrence using OGFs. Typically when the coefficients of the recurrence are polynomials in the index n, then the relationship constraining the generating function is a differential equation. Recall that the Quicksort recurrence is:

$$nC_n = n(n + 1) + 2 \sum_{k=1}^{n} C_{k-1}, \text{ for } n \geq 1 \text{ with } C_0 = 0.$$

Define $C(z) = \sum_{n \geq 0} C_n z^n$.

1. Multiply both sides of the recurrence by z^n and sum over the index n to obtain a functional relation involving $C(z)$, $C'(z)$, and z of the form:

$$C'(z) + P(z)C(z) = Q(z),$$

where $P(z)$ and $Q(z)$ are functions of z.

2. This differential equation can be solved by multiplying both sides by the "integrating factor" $e^{\int_0^z P(x) \mathrm{d}x}$. Notice that multiplying by this factor yields:

$$\begin{aligned} C'(z)e^{\int_0^z P(x)\mathrm{d}x} + P(z)C(z)e^{\int_0^z P(x)\mathrm{d}x} &= Q(z)e^{\int_0^z P(x)\mathrm{d}x} \\ \left(C(z)e^{\int_0^z P(x)\mathrm{d}x}\right)' &= Q(z)e^{\int_0^z P(x)\mathrm{d}x} \end{aligned}$$

We can now express $C(z)$ as

$$C(z) = e^{-\int_0^z P(x)\mathrm{d}x} \int Q(z)e^{\int_0^z P(x)\mathrm{d}x} \mathrm{d}z.$$

Solution

As suggested, multiplying both sides of the recurrence by z^n and then summing over the index n yields

$$\sum_{n=0}^{\infty} nC_n z^n = \sum_{n=0}^{\infty} n(n + 1)z^n + 2 \sum_{n=0}^{\infty} \left(\sum_{k=1}^{n} C_{k-1} \right) z^n. \tag{1}$$

Now $\sum_{n\geq 0} nC_n z^n = zC'(z)$. Note that

$$\begin{aligned}\sum_{n=0}^{\infty} n(n+1)z^n &= 2\sum_{n=0}^{\infty}\binom{n+1}{2}z^n\\ &= \frac{2}{z}\sum_{n=0}^{\infty}\binom{n+1}{2}z^{n+1}\\ &= \frac{2}{z}\sum_{n=2}^{\infty}\binom{n}{2}z^n\\ &= \frac{2}{z}\frac{z^2}{(1-z)^3}\\ &= \frac{2z}{(1-z)^3}.\end{aligned}$$

Finally, note that

$$\begin{aligned}\sum_{n=0}^{\infty}\left(\sum_{k=1}^{n} C_{k-1}\right)z^n &= \sum_{n=0}^{\infty} z^n \sum_{n=0}^{\infty} C_n z^{n+1}\\ &= \frac{zC(z)}{1-z}.\end{aligned}$$

Hence the functional relation that we are seeking is:

$$C'(z) - \frac{2}{1-z}C(z) = \frac{2}{(1-z)^3}.$$

We can solve this differential equation ourself, or use a computer algebra system, which solves it for us with ease. For example, if we type `c'(z) - 2/(1-z)*c(z) = 2/(1-z)^3, c(0) = 0` into *WolframAlpha* we get the result:

$$\frac{2i(\pi + i\ln(z-1))}{(z-1)^2}$$

By realizing that $\ln(z-1) = \ln(1-z) + \pi i$ we can rewrite the solution in the much nicer form

$$\frac{2}{(1-z)^2}\ln\frac{1}{1-z}.$$

In the good ol' days when such tools were not yet available and we had to solve such differential equations by hand we proceeded as follows.
The integrating factor in our case is

$$e^{-2\int_{x=0}^{z}\frac{1}{1-z}\mathrm{d}x} = e^{2\ln(1-z)} = (1-z)^2.$$

Hence $C(z)$ is given by

$$C(z) = \frac{1}{(1-z)^2}\int \frac{2}{1-z}\mathrm{d}z = \frac{2}{(1-z)^2}\ln\frac{1}{1-z}.$$

Using the fact that $\frac{z}{(1-z)^2}\ln\frac{1}{1-z} = \sum_{n\geq 0} n(H_n - 1)z^n$, we obtain:

$$[z^n]\frac{2}{(1-z)^2}\ln\frac{1}{1-z} = 2(n+1)(H_{n+1}-1).$$

Problem H15 (15 credits)
Find $[z^n]$ for each of the following OGFs.

$$\frac{1}{(1-3z)^4}, \qquad (1-z)^2 \ln\frac{1}{1-z}, \qquad \frac{1}{(1-2z^2)^2}$$

Solution

1. We may write $\frac{1}{(1-3z)^4}$ as
$$\frac{1}{(1-3z)^4} = \sum_{n=0}^{\infty} \binom{n+3}{3} (3z)^n.$$
This follows from the table of OGFs that we discussed in class. Thus the coefficient of z^n is $\binom{n+3}{3} \cdot 3^n$.

2. We may write the next function as:
$$(1-z)^2 \ln\frac{1}{1-z} = (1-z)^3 \cdot \frac{1}{1-z} \ln\frac{1}{1-z} = \sum_{n=0}^{\infty} \binom{3}{n} (-z)^n \cdot \sum_{n=0}^{\infty} H_n z^n,$$
where $H_0 = 0$. Now $[z^n]$ for this function is $\sum_{k=0}^{n} \binom{3}{k} (-1)^k H_{n-k}$.

3. The last function may be written as:
$$\frac{1}{(1-2z^2)^2} = \sum_{n=0}^{\infty} \binom{n+1}{1} (2z^2)^n = \sum_{n=0}^{\infty} 2^n \cdot (n+1) z^{2n}.$$
Thus for odd n, the coefficient of z^n is 0. For even n, the coefficient of z^n is $2^{n/2}(\frac{n}{2}+1)$.

Problem H16 (15 credits)
Find the ordinary generating function (OGF) for each of the following sequences:

(1) $(k2^{k+1})_{k\geq 0}$ (2) $s_0 = 0$ and $s_k = \frac{1}{k}$ for $k \geq 1$ (3) $(H_k)_{k\geq 1}$
(4) $(kH_k)_{k\geq 1}$ (5) $(k^3)_{k\geq 2}$

Solution

1. The ordinary generating function for the sequence $\{k2^{k+1}\}_{k\geq 0}$ is:
$$\sum_{k\geq 0} k2^{k+1} z^k = \sum_{k\geq 0} 2k(2z)^k = 2z \sum_{k\geq 1} 2k(2z)^{k-1} = 2z \left(\sum_{k\geq 0} (2z)^k \right)'.$$
Now the last term may be rewritten as:
$$2z \left(\frac{1}{1-2z} \right)' = \frac{4z}{(1-2z)^2}.$$

2. Start with the generating function $G(z) = \sum_{k=0}^{\infty} z^k = \frac{1}{1-z}$. Integrating both sides we obtain:

$$\begin{aligned}\int_0^z \sum_{k=0}^{\infty} z^k \mathrm{d}z &= \int_0^z \frac{1}{1-z}\mathrm{d}z \\ &= -\ln(1-z)\,|_0^z \\ &= \ln\frac{1}{1-z}.\end{aligned}$$

The series itself can be integrated term-by-term. This is valid if z is in the radius of convergence of the power series, and we can make this assumption because the value of z itself is not of interest to us. This then gives us:

$$\ln\frac{1}{1-z} = \sum_{k=0}^{\infty}\frac{z^{k+1}}{k+1} = \sum_{k=1}^{\infty}\frac{z^k}{k}.$$

Since $s_0 = 0$, the OGF of the sequence is $\ln(1-z)^{-1}$.

3. Let s_k be as defined before: $s_0 = 0$ and $s_k = 1/k$ for $k \geq 1$. Also define $H_0 = 0$. Then the OGF for $\{H_k\}_{k\geq 0}$ is

$$\begin{aligned}\sum_{k=0}^{\infty} H_k z^k &= \sum_{k=0}^{\infty}\left(H_0 + 1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{k}\right) z^k \\ &= \sum_{k=0}^{\infty}\left(\sum_{i=0}^{k} 1\cdot s_{k-i}\right) z^k \\ &= \sum_{k=0}^{\infty} z^k \sum_{k=0}^{\infty} s_k z^k \\ &= \frac{1}{1-z}\ln\frac{1}{1-z}.\end{aligned}$$

4. We start with the identity obtained in the last exercise: $\sum_{k\geq 0} H_k z^k = \frac{1}{1-z}\ln\frac{1}{1-z}$. Differentiating both sides and multiplying by z, we obtain:

$$\sum_{k=1}^{\infty} kH_k z^k = \frac{z}{(1-z)^2}\left(1 + \ln\frac{1}{1-z}\right).$$

Now the index k in power series on the left hand side may be written so that it ranges from 0 to ∞. This shows that the expression on the left hand side is the closed-form of the generating function that we are seeking.

5. We start with the fact that $\sum_{k\geq 0} z^k = \frac{1}{1-z}$. Differentiating both sides and using the fact that a power series can be termwise differentiated w.r.t z if z is in the radius of convergence, we obtain:

$$\sum_{k=1}^{\infty} kz^{k-1} = \ln\frac{1}{1-z}. \tag{2}$$

Multiplying both sides by z and differentiating again, we obtain:

$$\sum_{k=1}^{\infty} k^2 z^{k-1} = \ln\frac{1}{1-z} + \frac{z}{1-z}. \tag{3}$$

We again multiply both sides by z and differentiate, to obtain:

$$\sum_{k=1}^{\infty} k^3 z^{k-1} = \ln \frac{1}{1-z} + \frac{3z}{1-z} + \frac{z^2}{(1-z)^2}. \tag{4}$$

Again multiplying the above by z, we obtain:

$$\sum_{k=1}^{\infty} k^3 z^{k} = z \ln \frac{1}{1-z} + \frac{3z^2}{1-z} + \frac{z^3}{(1-z)^2}. \tag{5}$$

Now the generating function for our sequence is $\sum_{k\geq 2} k^3 z^k$ and this equals the right hand side of (5) minus z, that is:

$$z \ln \frac{1}{1-z} + \frac{3z^2}{1-z} + \frac{z^3}{(1-z)^2} - z.$$

Problem H17 (10 credits)
Solve the recurrences:

1. $a_n = 5a_{n-1} - 8a_{n-2} + 4a_{n-3}$ for $n > 2$ with $a_0 = 1$, $a_1 = 2$, and $a_2 = 4$.
2. $a_n = 3a_{n-1} - 3a_{n-2} + a_{n-3}$ for $n > 2$ with $a_0 = a_1 = 0$ and $a_2 = 1$.

Hint: You might need to use:

$$\frac{1}{(1-z)^{m+1}} = \sum_{n=0}^{\infty} \binom{n+m}{m} z^n.$$

Solution

1. As in the exercise, we have $g(z) = 1 - 5z + 8z^2 - 4z^3 = (1-z)(1-2z)^2$. As before, we use the initial conditions to express $f(z)$:

$$\begin{aligned} f(z) &= g(z) \sum_{n=0}^{2} a_n z^n \pmod{z^3} \\ &= (1 - 5z + 8z^2 - 4z^3)(1 + 2z + 4z^2) \pmod{z^3} \\ &= 1 - 3z + 2z^2 \\ &= (1-z)(1-2z). \end{aligned}$$

   Now $a(z) = 1/(1-2z) = \sum_{n\geq 0}(2z)^n$, so that $a_n = 2^n$ for all $n \geq 0$.

2. Compute $g(z) = 1 - 3z + 3z^2 - z^3 = (1-z)^3$. Using the initial conditions, we can write $f(z)$ as:

$$\begin{aligned} f(z) &= (1-z)^3 z^2 \pmod{z^3} \\ &= z^2. \end{aligned}$$

   Thus $a(z) = z^2/(1-z)^3$. We can write this as

$$\frac{z^2}{(1-z)^3} = \frac{\alpha}{1-z} + \frac{\beta}{(1-z)^2} + \frac{\gamma}{(1-z)^3}.$$

Substituting $z = 0, -1, -2$, we obtain the following system of equations:

$$\alpha + \beta + \gamma = 0$$
$$4\alpha + 2\beta + \gamma = 1$$
$$9\alpha + 3\beta + \gamma = 4$$

from which we obtain: $\alpha = 1$, $\beta = -2$ and $\gamma = 1$. Thus

$$\begin{aligned} a(z) &= \frac{1}{1-z} - \frac{2}{(1-z)^2} + \frac{1}{(1-z)^3} \\ &= \sum_{n=0}^{\infty} z^n - 2\sum_{n=0}^{\infty} \binom{n+1}{1} z^n + \sum_{n=0}^{\infty} \binom{n+2}{2} z^n. \end{aligned}$$

Hence $a_n = 1 - 2\binom{n+1}{1} + \binom{n+2}{2}$ for $n \geq 3$. This simplifies to $\binom{n}{2}$.

1

1 1

1 2 1

1 3 3 1

1 4 6 4 1

1 5 10 10 5 1

1 6 15 20 15 6 1

1 7 21 35 35 21 7 1

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