Analysis of Algorithms, WS 2020
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Exercise Sheet 07

Problem T16

This tutorial is geared towards solving recurrence relations using generating functions. Here is an observation: If a_n satisfies the recurrence

$$a_n = x_1 a_{n-1} + x_2 a_{n-2} + \cdots + x_t a_{n-t}$$

for $n \geq t$, then the generating function $a(z) = \sum_{n\geq 0} a_n z^n$ is a rational function $a(z) = f(z)/g(z)$, where the denominator polynomial is $g(z) = 1 - x_1 z - x_2 z^2 - \cdots - x_t z^t$ and the numerator polynomial is determined by the intial values $a_0, a_1, \ldots, a_{t-1}$ and has degree at most $t-1$. In fact, $f(z)$ may be written as:

$$f(z) = g(z) \sum_{n=0}^{t-1} a_n z^n \left(\bmod\ z^t\right).$$

Solve the following recurrence relations:

1. $a_n = 2a_{n-1} + a_{n-2} - 2a_{n-3}$ for $n > 2$ with $a_0 = 0$ and $a_1 = a_2 = 1$.
2. $a_n = 5a_{n-1} - 8a_{n-2} + 4a_{n-3}$ for $n > 2$ with $a_0 = 0$, $a_1 = 1$, and $a_2 = 4$.

Problem T17

In this problem, we will solve the *Quicksort* recurrence using OGFs. Typically when the coefficients of the recurrence are polynomials in the index n, then the relationship constraining the generating function is a differential equation. Recall that the Quicksort recurrence is:

$$nC_n = n(n+1) + 2\sum_{k=1}^{n} C_{k-1}, \text{ for } n \geq 1 \text{ with } C_0 = 0.$$

Define $C(z) = \sum_{n\geq 0} C_n z^n$.

1. Multiply both sides of the recurrence by z^n and sum over the index n to obtain a functional relation involving $C(z)$, $C'(z)$, and z of the form:

   $$C'(z) + P(z)C(z) = Q(z),$$

   where $P(z)$ and $Q(z)$ are functions of z.

2. This differential equation can be solved by multiplying both sides by the "integrating factor" $e^{\int_0^z P(x)\mathrm{d}x}$. Notice that multiplying by this factor yields:

   $$C'(z)e^{\int_0^z P(x)\mathrm{d}x} + P(z)C(z)e^{\int_0^z P(x)\mathrm{d}x} = Q(z)e^{\int_0^z P(x)\mathrm{d}x}$$
   $$\left(C(z)e^{\int_0^z P(x)\mathrm{d}x}\right)' = Q(z)e^{\int_0^z P(x)\mathrm{d}x}$$

   We can now express $C(z)$ as

   $$C(z) = e^{-\int_0^z P(x)\mathrm{d}x} \int Q(z)e^{\int_0^z P(x)\mathrm{d}x}\mathrm{d}z.$$

Problem H15 (15 credits)
Find $[z^n]$ for each of the following OGFs.

$$\frac{1}{(1-3z)^4}, \qquad (1-z)^2 \ln \frac{1}{1-z}, \qquad \frac{1}{(1-2z^2)^2}$$

Problem H16 (15 credits)
Find the ordinary generating function (OGF) for each of the following sequences:

(1) $(k2^{k+1})_{k\geq 0}$ (2) $s_0 = 0$ and $s_k = \frac{1}{k}$ for $k \geq 1$ (3) $(H_k)_{k\geq 1}$
(4) $(kH_k)_{k\geq 1}$ (5) $(k^3)_{k\geq 2}$

Problem H17 (10 credits)
Solve the recurrences:

1. $a_n = 5a_{n-1} - 8a_{n-2} + 4a_{n-3}$ for $n > 2$ with $a_0 = 1$, $a_1 = 2$, and $a_2 = 4$.
2. $a_n = 3a_{n-1} - 3a_{n-2} + a_{n-3}$ for $n > 2$ with $a_0 = a_1 = 0$ and $a_2 = 1$.

Hint: You might need to use:

$$\frac{1}{(1-z)^{m+1}} = \sum_{n=0}^{\infty} \binom{n+m}{m} z^n.$$

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1 1

1 2 1

1 3 3 1

1 4 6 4 1

1 5 10 10 5 1

1 6 15 20 15 6 1

1 7 21 35 35 21 7 1

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