Analysis of Algorithms, WS 2020
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Exercise Sheet with solutions 06

Problem T14

Compute the generating functions of the following series:

(a) $a_n = 2^n + 3^n$ (b) $b_n = (n+1)2^{n+1}$ (c) $c_n = \alpha^n \binom{k}{n}$
(d) $d_n = n - 1$ (e) $e_n = (n+1)^2$

Solution

(a) The generating function of (α^n) is $\sum_{n\geq 0} \alpha^n z^n$, which yields $\frac{1}{1-\alpha z}$ in closed form. The generating function of $a_n = 2^n + 3^n$ is thus simply $\frac{1}{1-2z} + \frac{1}{1-3z}$.

(b) We start with (2^n) and $\frac{1}{1-2z}$. Derivating yields $b_n = (n+1)2^{n+1}$ with generating function $\frac{2}{(1-2z)^2}$.

(c) The series $\binom{k}{n}$ has the generating function $(1+z)^k$. Scaling with α results in $c_n = \alpha^n \binom{k}{n}$ with corresponding generating function $(1+\alpha z)^k$.

(d) We already know that the series $(n+1) = 1, 2, 3, 4, \ldots$ belongs to the generating function $\frac{1}{(1-z)^2}$. In order to obtain $d_n = -1, 0, 1, 2, 3, \ldots$, we first shift this twice to the right. This yields $0, 0, 1, 2, 3, 4, \ldots$ with generating function $\frac{z^2}{(1-z)^2}$. Now we subtract $1, 0, 0, \ldots$ and obtain d_n with generating function $\frac{z^2}{(1-z)^2} - 1$.

(e) Recall that $(n+1) = 1, 2, 3, 4, \ldots$ has the generating function $\frac{1}{(1-z)^2}$. We shift to the right and obtain (n) as well as $\frac{z}{(1-z)^2}$. Derivating yields the desired series $e_n = (n+1)^2$ with generating function $\frac{z+1}{(1-z)^3}$.

Problem T15

Compute:

$$\text{(a) } [z^n]\frac{1}{1+2z} \qquad \text{(b) } [z^n]\frac{z+1}{z-1} \qquad \text{(c) } [z^n]\left(\frac{z+1}{z-1}\right)^2 \qquad \text{(d) } [z^n]\frac{1}{\sqrt[3]{5+z}}$$

Solution

(a) We know that $\sum_{n\geq 0} \alpha^n z^n$, yields $\frac{1}{1-\alpha z}$. So $[z^n]\frac{1}{1+2z} = (-2)^n$.

(b) We use that

$$\frac{n+1}{n-1} = 1 + 2\frac{1}{1-n}$$

and get $A(z) + 2B(z)$ with

$$[z^n]A(z) = (n = 0)$$

and

$$[z^n]B(z) = 1$$

to obtain $[z^n]\frac{z+1}{z-1} = (n = 0) + 2$.

(c) The convolution rule yields

$$A(z)^2 = \sum_{n=0}^{\infty} \Big(\sum_{k=0}^{n} a_k a_{n-k} \Big) z^n$$

We use our solution for a_n from (b) to obtain For $n \neq 0$

$$\sum_{k=0}^{n} a_k a_{n-k} = \Big(\sum_{k=1}^{n-1} a_k a_{n-k} \Big) + a_0 \cdot a_n + a_n \cdot a_0 = 4(n-1) + 4 = 4n$$

and for $n = 0$

$$\sum_{k=0}^{n} a_k a_{n-k} = 1$$

Which yields $[z^n]\left(\frac{z+1}{z-1}\right)^2 = 4n + (n = 0)$.

(d) We start with

$$\frac{1}{\sqrt[3]{5+z}} = \frac{1}{\sqrt[3]{5}} \frac{1}{\sqrt[3]{1+z/5}}$$

We use that

$$[z^n](1+z)^r = \binom{r}{n}$$

Finally we use scaling with 1/5 to obtain

$$[z^n]\frac{1}{\sqrt[3]{5+z}} = \frac{1}{\sqrt[3]{5}} \binom{-\frac{1}{3}}{n} 5^{-n} .$$

Problem H13 (15 credits)
Let $A(z)$ and $B(z)$ be the OGFs of two series a_n and b_n.
The convolution $c_n = (a_n)_{n=0}^{\infty} * (b_n)_{n=0}^{\infty}$ of a_n and b_n is defined as

$$c_n = \sum_{k=0}^{n} a_k b_{n-k}.$$

For example,

$$(n)_{n=0}^{\infty} * (3^n)_{n=0}^{\infty} = \Big(\sum_{k=0}^{n} k 3^{n-k} \Big)_{n=0}^{\infty}.$$

Prove that the OGF of the convolution of a_n and b_n is $A(z)B(z)$.

Solution
Let $A(z) = \sum_{n=0}^{\infty} a_n z^n$ and $B(z) = \sum_{n=0}^{\infty} b_n z^n$. Then

$$A(z)B(z) = \sum_{n=0}^{\infty} a_n z^n \sum_{m=0}^{\infty} b_m z^m = \sum_{n=0}^{\infty} a_n \sum_{m=0}^{\infty} b_k z^{n+m} = \sum_{n=0}^{\infty} a_n \sum_{k=0}^{\infty} b_{k-n} z^k$$
$$= \sum_{k=0}^{\infty} \sum_{n=0}^{\infty} a_n b_{k-n} z^k = \sum_{k=0}^{\infty} \sum_{n=0}^{k} a_n b_{k-n} z^k = \sum_{k=0}^{\infty} c_k z^k = C(z)$$

Problem H14 (15 credits)
Solve this recurrence using generating functions:

$$a_n = 2a_{n-1} + 3a_{n-2}$$

and $a_0 = 0$, $a_1 = 2$.

Solution
We start with $S(z) = \sum_{n\geq 0} a_n z^n$. Since the values a_0 and a_1 are given, we rewrite this as

$$S(z) = a_0 + a_1 z + \sum_{n\geq 2} a_n z^n.$$

Using the recursive definition, this yields

$$S(z) = a_0 + a_1 z + \sum_{n\geq 2}(2a_{n-1} + 3a_{n-2})z^n = a_0 + a_1 z + \sum_{n\geq 2} 2a_{n-1}z^n + \sum_{n\geq 2} 3a_{n-2}z^n.$$

Shifting results in

$$\begin{aligned} S(z) &= a_0 + a_1 z + z\sum_{n\geq 1} 2a_n z^{n+1} + \sum_{n\geq 0} 3a_n z^{n+2} \\ &= a_0 + a_1 z - 2a_0 z + 2z\sum_{n\geq 0} a_n z^n + 3z^2 \sum_{n\geq 0} a_n z^n \\ &= 2z + 2zS(z) + 3z^2 S(z) \end{aligned}$$

This implies

$$S(z) = \frac{2z}{1-2z-3z^2} = \frac{2z}{(1+z)(1-3z)}$$

Now, we need to find a and b such that

$$S(z) = \frac{2z}{(1+z)(1-3z)} = \frac{a}{1+z} + \frac{b}{1-3z}.$$

Setting $z = 0$ and $z = 1$ implies $a + b = 0$ as well as $-2a + 2b = 2$, and thus $a = -1/2$ and $b = 1/2$. We obtain

$$S(z) = -\frac{1/2}{1+z} + \frac{1/2}{1-3z} = -\frac{1}{2}\sum_{n\geq 0}(-1)^n z^n + \frac{1}{2}\sum_{n\geq 0} 3^n z^n$$

Thus, we have

$$a_n = \frac{1}{2}3^n - \frac{1}{2}(-1)^n.$$