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Theoretical Computer Science
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Exercise 12
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Exercise for Analysis of Algorithms

Exercise E1

Determine a recurrence relation for the number B_n of comparisons, when the following algorithm is used to find an element x contained in an array $a[1], \ldots, a[n]$. We assume that the array is sorted in increasing order and contains x.
Then compute B_3.

Algorithm: Binary Search with randomly chosen pivot element

1. Choose randomly and with uniform probability an $i \in \{1, \ldots, n\}$.
2. If $a[i] = x$, output i and halt.
3. Continue recursively on the left ($x < a[i]$) or right ($x > a[i]$) subarray.

Solution:
We always have to make at least one comparison, and each pivot is chosen with probability $1/n$. If we choose element k, we get two remaining intervals of size $k-1$ and $n-k$, which are chosen with $(k-1)/n$ and $(n-k)/n$ respectively. Therefore we get

$$B_n = 1 + \frac{1}{n}\sum_{k=1}^{n}\Big(\frac{k-1}{n}B_{k-1} + \frac{n-k}{n}B_{n-k}\Big)$$

$$B_n = 1 + \frac{2}{n^2}\sum_{k=0}^{n-1} kB_k$$

From that we can easily calculate $B_1 = 1$, $B_2 = \frac{3}{2}$ and $B_3 = \frac{17}{9}$.

Exercise E2

$$A(z) = \frac{\sqrt{1-z^7}}{2z^2-3z+1} \qquad B(z) = \frac{1-z^2}{e^{z+3z^2}} \qquad C(z) = z^5 + 3z^2(z^3 + z^2 + 8)$$

Order the coefficients of the sequences a_n, b_n, and c_n in increasing order by their asymptotic growth and give a proof.

Solution:
$A(z)$ has a dominant singularity at $\frac{1}{2}$, which means it as an exponential growth of 2^n. $B(z)$ does not have any singularities and has therefore subexponential growth and $[z^n]C(z) = 4(n = 5) + 3(z = 4) + 24(z = 2)$ is always zero except for finite many exceptions. From that follows $C(z) < B(z) < A(z)$.

Exercise E3

Consider the number B_n of $2-3$–trees (each inner node has exactly two or three children) with n leaves. Does B_n grow asymptotically slower or faster than 5^n?

Hint: The following `maxima` output, which finds roots of equations, might help you to answer this question: $solve(T^3 + T^2 - T + z = 0, T)$:

$$\left[T = \left(-\frac{\sqrt{3}\,i}{2} - \frac{1}{2}\right)\left(\frac{\sqrt{27\,z^2+22\,z-5}}{6\,\sqrt{3}} - \frac{27\,z+11}{54}\right)^{\frac{1}{3}} + \frac{4\left(\frac{\sqrt{3}\,i}{2} - \frac{1}{2}\right)}{9\left(\frac{\sqrt{27\,z^2+22\,z-5}}{6\,\sqrt{3}} - \frac{27\,z+11}{54}\right)^{\frac{1}{3}}} - \frac{1}{3},\right.$$

$$T = \left(\frac{\sqrt{3}\,i}{2} - \frac{1}{2}\right)\left(\frac{\sqrt{27\,z^2+22\,z-5}}{6\,\sqrt{3}} - \frac{27\,z+11}{54}\right)^{\frac{1}{3}} + \frac{4\left(-\frac{\sqrt{3}\,i}{2} - \frac{1}{2}\right)}{9\left(\frac{\sqrt{27\,z^2+22\,z-5}}{6\,\sqrt{3}} - \frac{27\,z+11}{54}\right)^{\frac{1}{3}}} - \frac{1}{3},$$

$$\left. T = \left(\frac{\sqrt{27\,z^2+22\,z-5}}{6\,\sqrt{3}} - \frac{27\,z+11}{54}\right)^{\frac{1}{3}} + \frac{4}{9\left(\frac{\sqrt{27\,z^2+22\,z-5}}{6\,\sqrt{3}} - \frac{27\,z+11}{54}\right)^{\frac{1}{3}}} - \frac{1}{3}\right]$$

Solution:

The symbolic method yields

$$B(z) = z + B(z)^2 + B(z)^3.$$

From `maxima` we see that $B(z)$ has a singularity in the root for

$$27z^2 + 22z - 5 = 0$$

We can factor this to

$$(z+1)(27z-5) = 0$$

and find the dominant singularity at $\frac{5}{27}$, which means the exponential growth is $(\frac{27}{5})^n > 5^n$.

Exercise E4

Determine g_n up to an additive error of $O(4^n)$ for

$$G(z) = \sum_{n=0}^{\infty} g_n z^n = \frac{15z^2 + 8z + 1}{15z^2 - 8z + 1}.$$

Solution:

We have

$$G(z) = \frac{15z^2 + 8z + 1}{(3z-1)(5z-1)},$$

the singularities are at $\frac{1}{3}$ and $\frac{1}{5}$, and both are poles of first order. Because of

$$G(z) \sim \frac{8}{5}\frac{1}{(z - \frac{1}{5})} \text{ for } z \to \frac{1}{5}$$

the difference between g_n and $[z^n]\frac{8}{1-5z}$ is at most $O(3^n)$. Therefore $g_n = \frac{8}{5} \cdot 5^n + O(3^n)$.