

Analysis of Algorithms

Recall from class that if a complex-valued function $f(z)$ is analytic at 0 and R is the absolute value of the singularity nearest to the origin then the coefficient $f_n = [z^n]f(z)$ satisfies:

$$f_n \asymp \left(\frac{1}{R}\right)^n.$$

Exercise 11-1

Sort the following generating functions *within one minute* by their exponential growth!

1. $A(z) = 1/\sqrt{1-z/2}$
2. $B(z) = (1+z)/(1-z)$
3. $C(z) = \frac{1}{1-e^{-z-1/3}}$

Solution: We only need to sort them by the absolute value of the dominant singularities. $A_n \asymp 1/2^n$, $B_n \asymp 1$, $C_n \asymp 3^n$.

Exercise 11-2

An algorithm is given an array of length $n \geq 0$ and, if $n \geq 2$, for each $1 \leq k \leq n$ calls itself on some random subarray of length k with probability $\frac{1}{2}$. Compute the exponential growth of the running time of this algorithm.

Solution:

Let t_n be the expected number of calls of the algorithm for an array of length n . Then $t_0 = t_1 = 1$ and $t_n = 1 + \frac{1}{2} \sum_{k=1}^n t_k$ for $n > 1$. To get an equation that holds for all n , we let

$$t_n = 1 + \frac{1}{2} \sum_{k=1}^n t_k - \frac{1}{2}(n=1).$$

We make yet another modification so that in the equation above so that we may sum from $k=0$:

$$t_n = 1 + \frac{1}{2} \sum_{k=0}^n t_k - \frac{1}{2}(n=1) - \frac{1}{2}.$$

Define $T(z) = \sum_{n=0}^{\infty} t_n z^n$. Multiplying both sides by z^n and summing over n , we obtain:

$$T(z) = \frac{1}{1-z} + \frac{1}{2(1-z)}T(z) - \frac{1}{2}z + \frac{1}{2(1-z)}$$

which gives the following functional relation: $T(z) = \frac{z^2-z+3}{1-2z}$. Since the dominant singularity is located at $z = \frac{1}{2}$, we get an asymptotic running time of $\asymp 2^n$.

Exercise 11-3

Determine $[z^n]_{2-e^z}$ up to an additive error of $O(12^{-n})!$

Solution: We first determine the dominant singularity, which is $\ln 2$. Since, by L'Hôpital,

$$\lim_{z \rightarrow \ln 2} \frac{\ln 2 - z}{2 - e^z} = \frac{1}{2},$$

we have

$$\frac{\ln 2 - z}{\ln 2 - z} \cdot \frac{1}{2 - e^z} \sim \frac{1}{2} \cdot \frac{1}{\ln 2 - z} = \frac{1}{2 \ln 2} \cdot \frac{1}{1 - (1/\ln 2)z} = \frac{1}{2 \ln 2} \sum_{n=0}^{\infty} \left(\frac{1}{\ln 2}\right)^n z^n.$$

We now look for the next singularities (ordered by their distance from the origin). These are $\ln 2 \pm 2\pi i$. By L'Hôpital

$$\lim_{z \rightarrow \ln 2 \pm 2\pi i} \frac{z - \ln 2 \mp 2\pi i}{2 - e^z} = -\frac{1}{2}$$

and therefore

$$S(z) \sim -\frac{1}{2} \frac{1}{z - \ln 2 \mp 2\pi i} = \frac{1}{2} \frac{1}{\ln 2 \mp 2\pi i} \frac{1}{1 - \frac{z}{\ln 2 \mp 2\pi i}}$$

for $z \rightarrow \ln 2 \pm 2\pi i$. We can now use the Theorem of the lecture to determine the coefficients:

$$\begin{aligned} [z^n]S(z) &= \frac{1}{2} \left(\left(\frac{1}{\ln 2}\right)^{n+1} + \left(\frac{1}{\ln 2 + 2\pi i}\right)^{n+1} + \left(\frac{1}{\ln 2 - 2\pi i}\right)^{n+1} \right) + O(r)^{-n} \\ &= \frac{1}{2} \left(\left(\frac{1}{\ln 2}\right)^{n+1} + r^{n+1} (e^{i\phi(n+1)} + e^{-i\phi(n+1)}) \right) + O(r)^{-n} \\ &= \frac{1}{2} \left(\left(\frac{1}{\ln 2}\right)^{n+1} + r^{n+1} \cos(\phi(n+1)) \right) + O(r)^{-n} \end{aligned}$$

with $r = 1/\sqrt{\ln^2 2 + 4\pi^2} \approx 12.58547409739904$, $\phi = \arctan(\frac{2\pi}{\ln 2})$.

Homework Assignment 11-1 (10 points)

Determine the exponential growth of $[z^n]G(z)$, where

- $G(z) = z^2/(1 - z - z^2)$,
- $G(z) = \sqrt{1 + 2z} - \sqrt{2 + 2z - 4z^2}/\sqrt{3}$,
- $G(z) = \ln(1 + \sin(z))/\ln(1 + \cos(z))$.

Solution:

- We solve $1 - z - z^2 = 0$, which yields $z = -\frac{1}{2} \pm \frac{1}{2}\sqrt{5}$. The dominant singularity is at $\phi = \frac{1}{2}\sqrt{5} - \frac{1}{2} \approx 0.6181$. Hence, $[z^n]G(z) \asymp \phi^{-n} \approx 1.6181^n$.
- The dominant singularity is at $-\frac{1}{2}$. Therefore, $[z^n]G(z) \asymp 2^n$.
- Using the rules for the logarithm, we get $G(z) = \ln(\sin(z) - \cos(z))$. The dominant singularity is at $\pi/4$. Therefore, $[z^n]G(z) \asymp (4/\pi)^n \approx 1.2732^n$.

Homework Assignment 11-2 (10 points)

$$A(z) = \frac{\sqrt{1-z^7}}{2z^2-3z+1} \quad B(z) = \frac{1-z^2}{e^{z+3z^2}} \quad C(z) = z^5 + 3z^2(z^3 + z^2 + 8)$$

Sort the sequences a_n , b_n , and c_n by their exponential growth.

Solution:

We have $c_n = 0$ for almost all n . The sequence b_n growth subexponentially, because $B(z)$ lacks a singularity. The dominant singularity of $A(z)$ is at $\frac{1}{2}$, i.e., $[z^n]A(z) \asymp 2^n$.

Homework Assignment 11-3 (10 Points)

Determine $g_n = [z^n]G(z)$ up to an additive error of $O(4^n)$, where

$$G(z) = \sum_{n=0}^{\infty} g_n z^n = \frac{15z^2 + 8z + 1}{15z^2 - 8z + 1}.$$

Solution: We have

$$G(z) = \frac{15z^2 + 8z + 1}{(3z - 1)(5z - 1)}.$$

Its singularities are at $\frac{1}{3}$ and $\frac{1}{5}$, both being poles of order 1. Since

$$G(z) \sim \frac{8}{1-5z} \text{ f\"ur } z \rightarrow \frac{1}{5},$$

we have $g_n - [z^n]\frac{8}{1-5z} = O((3+\epsilon)^n)$ for any $\epsilon > 0$. Therefore, $g_n = 8 \cdot 5^n + O(4^n)$.