RWTH Aachen **SS 2014**
Lehrgebiet Theoretische Informatik Tutorial 8
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Analysis of Algorithms

Problem 8-1

Find the ordinary generating function (OGF) for each of the following sequences:

(1) $\{k2^{k+1}\}_{k\geq 0}$ (2) $s_0 = 0$ and $s_k = \frac{1}{k}$ for $k \geq 1$ (3) $\{H_k\}_{k\geq 1}$
(4) $\{kH_k\}_{k\geq 1}$ (5) $\{k^3\}_{k\geq 2}$

Solution

1. The ordinary generating function for the sequence $\{k2^{k+1}\}_{k\geq 0}$ is:

$$\sum_{k\geq 0} k2^{k+1}z^k = \sum_{k\geq 0} 2k(2z)^k = 2z\sum_{k\geq 1} 2k(2z)^{k-1} = 2z\left(\sum_{k\geq 0}(2z)^k\right)'.$$

Now the last term may be rewritten as:

$$2z\left(\frac{1}{1-2z}\right)' = \frac{4z}{(1-2z)^2}.$$

2. Start with the generating function $G(z) = \sum_{k=0}^{\infty} z^k = \frac{1}{1-z}$. Integrating both sides we obtain:

$$\begin{aligned}\int_0^z \sum_{k=0}^{\infty} z^k \mathrm{d}z &= \int_0^z \frac{1}{1-z}\mathrm{d}z \\ &= -\ln(1-z)\,|_0^z \\ &= \ln\frac{1}{1-z}.\end{aligned}$$

The series itself can be integrated term-by-term. This is valid if z is in the radius of convergence of the power series, and we can make this assumption because the value of z itself is not of interest to us. This then gives us:

$$\ln\frac{1}{1-z} = \sum_{k=0}^{\infty}\frac{z^{k+1}}{k+1} = \sum_{k=1}^{\infty}\frac{z^k}{k}.$$

Since $s_0 = 0$, the OGF of the sequence is $\ln(1-z)^{-1}$.

3. Let s_k be as defined before: $s_0 = 0$ and $s_k = 1/k$ for $k \geq 1$. Also define $H_0 = 0$. Then the OGF for $\{H_k\}_{k\geq 0}$ is

$$\begin{aligned}
\sum_{k=0}^{\infty} H_k z^k &= \sum_{k=0}^{\infty} \left(H_0 + 1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{k} \right) z^k \\
&= \sum_{k=0}^{\infty} \left(\sum_{i=0}^{k} 1 \cdot s_{k-i} \right) z^k \\
&= \sum_{k=0}^{\infty} z^k \sum_{k=0}^{\infty} s_k z^k \\
&= \frac{1}{1-z} \ln \frac{1}{1-z}.
\end{aligned}$$

4. We start with the identity obtained in the last exercise: $\sum_{k\geq 0} H_k z^k = \frac{1}{1-z} \ln \frac{1}{1-z}$. Differentiating both sides and multiplying by z, we obtain:

$$\sum_{k=1}^{\infty} k H_k z^k = \frac{z}{(1-z)^2} \left(1 + \ln \frac{1}{1-z} \right).$$

Now the index k in power series on the left hand side may be written so that it ranges from 0 to ∞. This shows that the expression on the left hand side is the closed-form of the generating function that we are seeking.

5. We start with the fact that $\sum_{k\geq 0} z^k = \frac{1}{1-z}$. Differentiating both sides and using the fact that a power series can be termwise differentiated w.r.t z if z is in the radius of convergence, we obtain:

$$\sum_{k=1}^{\infty} k z^{k-1} = \ln \frac{1}{1-z}. \tag{1}$$

Multiplying both sides by z and differentiating again, we obtain:

$$\sum_{k=1}^{\infty} k^2 z^{k-1} = \ln \frac{1}{1-z} + \frac{z}{1-z}. \tag{2}$$

We again multiply both sides by z and differentiate, to obtain:

$$\sum_{k=1}^{\infty} k^3 z^{k-1} = \ln \frac{1}{1-z} + \frac{3z}{1-z} + \frac{z^2}{(1-z)^2}. \tag{3}$$

Again multiplying the above by z, we obtain:

$$\sum_{k=1}^{\infty} k^3 z^k = z \ln \frac{1}{1-z} + \frac{3z^2}{1-z} + \frac{z^3}{(1-z)^2}. \tag{4}$$

Now the generating function for our sequence is $\sum_{k\geq 2} k^3 z^k$ and this equals the right hand side of (4) minus z, that is:

$$z \ln \frac{1}{1-z} + \frac{3z^2}{1-z} + \frac{z^3}{(1-z)^2} - z.$$

Problem 8-2

```
1 func( int n ){
    int s = 0;
3   if ( n == 0 ) return 1;
    for( int i = 0; i < n; ++i )
5     s += func( i );
    return s;
7 }
```

Compute how often the 5th line of this program is executed using (ordinary) generating functions.

Solution

Let A_n denote the how often the 5th line is called on input n. We immediately obtain $A_0 = 0$ and $A_n = n + \sum_{i=0}^{n-1} A_i$. The corresponding generating function is thus

$$A(z) := \sum_{n=0}^{\infty} A_n z^n = \sum_{n=0}^{\infty} n z^n + \sum_{n=0}^{\infty} \left(\sum_{k=0}^{n-1} A_k \right) z^n.$$

Now the last term on the right hand side may be seen as the convolution of the series $\sum_{n=0}^{\infty} z^n$ and $\sum_{n=0}^{\infty} A_n z^{n+1}$. The convolution product is simply $zA(z)/(1-z)$ so that we may write:

$$A(z) = \frac{z}{(1-z)^2} + \frac{zA(z)}{1-z},$$

which implies:

$$A(z)(1-2z) = \frac{z}{1-z}$$

and therefore

$$A(z) = \frac{1}{(1-z)(1-2z)} z.$$

Let $\frac{1}{(1-z)(1-2z)} = \frac{a}{1-z} + \frac{b}{1-2z}$ for some a, b. Setting $z = 0$ implies $a + b = 1$ and $z = -1$ implies $a/2 + b/3 = 1/6$. We easily obtain $a = -1$ and $b = 2$ and therefore

$$A(z) = \left(-\frac{1}{1-z} + \frac{2}{1-2z} \right) z = -\sum_{n=0}^{\infty} z^{n+1} + \sum_{n=0}^{\infty} (2z)^{n+1} = \sum_{n=1}^{\infty} (2^n - 1) z^n.$$

Thus $A_0 = 0$ and $A_n = 2^n - 1$ for all $n \geq 1$.

Homework Assignment 8-1 (10 Points)

Solve this recurrence using generating functions:

$$a_n = 2a_{n-1} + 3a_{n-2}$$

and $a_0 = 0$, $a_1 = 2$.

Solution

We start with $S(z) = \sum_{n\geq 0} a_n z^n$. Since the values a_0 and a_1 are given, we rewrite this as

$$S(z) = a_0 + a_1 z + \sum_{n\geq 2} a_n z^n.$$

Using the recursive definition, this yields

$$S(z) = a_0 + a_1 z + \sum_{n\geq 2}(2a_{n-1} + 3a_{n-2})z^n = a_0 + a_1 z + \sum_{n\geq 2} 2a_{n-1}z^n + \sum_{n\geq 2} 3a_{n-2}z^n.$$

Shifting results in

$$\begin{aligned} S(z) &= a_0 + a_1 z + z\sum_{n\geq 1} 2a_n z^{n+1} + \sum_{n\geq 0} 3a_n z^{n+2} \\ &= a_0 + a_1 z - 2a_0 z + 2z\sum_{n\geq 0} a_n z^n + 3z^2 \sum_{n\geq 0} a_n z^n \\ &= 2z + 2zS(z) + 3z^2 S(z) \end{aligned}$$

This implies

$$S(z) = \frac{2z}{1 - 2z - 3z^2} = \frac{2z}{(1+z)(1-3z)}$$

Now, we need to find a and b such that

$$S(z) = \frac{2z}{(1+z)(1-3z)} = \frac{a}{1+z} + \frac{b}{1-3z}.$$

Setting $z = 0$ and $z = 1$ implies $a + b = 0$ as well as $-2a + 2b = 2$, and thus $a = -1/2$ and $b = 1/2$. We obtain

$$S(z) = -\frac{1/2}{1+z} + \frac{1/2}{1-3z} = -\frac{1}{2}\sum_{n\geq 0}(-1)^n z^n + \frac{1}{2}\sum_{n\geq 0} 3^n z^n$$

Thus, we have

$$a_n = \frac{1}{2}3^n - \frac{1}{2}(-1)^n.$$

Homework Assignment 8-2 (10 points)

Find $[z^n]$ for each of the following OGFs.

$$\tfrac{1}{(1-3z)^4}, \quad (1-z)^2 \ln \tfrac{1}{1-z}, \quad \tfrac{1}{(1-2z^2)^2}$$

Solution

1. We may write $\frac{1}{(1-3z)^4}$ as

$$\frac{1}{(1-3z)^4} = \sum_{n=0}^{\infty} \binom{n+3}{3} (3z)^n.$$

This follows from the table of OGFs that we discussed in class. Thus the coefficient of z^n is $\binom{n+3}{3} \cdot 3^n$.

2. We may write the next function as:

$$(1-z)^2 \ln \frac{1}{1-z} = (1-z)^3 \cdot \frac{1}{1-z} \ln \frac{1}{1-z} = \sum_{n=0}^{\infty} \binom{3}{n} (-z)^n \cdot \sum_{n=0}^{\infty} H_n z^n,$$

where $H_0 = 0$. Now $[z^n]$ for this function is $\sum_{k=0}^{n} \binom{3}{k} (-1)^k H_{n-k}$.

3. The last function may be written as:

$$\frac{1}{(1-2z^2)^2} = \sum_{n=0}^{\infty} \binom{n+1}{1} (2z^2)^n = \sum_{n=0}^{\infty} 2^n \cdot (n+1) z^{2n}.$$

Thus for odd n, the coefficient of z^n is 0. For even n, the coefficient of z^n is $2^{n/2}(\frac{n}{2}+1)$.