

## Analysis of Algorithms

### Problem 6-1

Use the operator technique to obtain a solution to the following recurrence:

$$a_{n+2} = (n+2)a_{n+1} - na_n + n,$$

with  $a_0 = 0 = a_1$ .

### Solution

We may rewrite the recurrence in operator notation as follows:

$$(\mathbf{E}^2 - (n+2)\mathbf{E} + n)a_n = n.$$

The polynomial  $\mathbf{E}^2 - (n+2)\mathbf{E} + n$  can be factored into  $(\mathbf{E} - 1)(\mathbf{E} - n)$ , using the identity  $\mathbf{E}n = (n+1)\mathbf{E}$ . We now let  $b_n = (\mathbf{E} - n)a_n$ , for  $n \geq 0$ , and solve for  $b_n$ . We have  $(\mathbf{E} - 1)b_n = n$ , which written in full, amounts to  $b_{n+1} = b_n + n$ , where  $n \geq 0$  and  $b_0 = a_1 = 0$ . This gives

$$b_n = b_0 + \frac{n(n-1)}{2} = \frac{n(n-1)}{2}.$$

We now solve for  $a_n$ . We have  $(\mathbf{E} - n)a_n = b_n$  and this may be written out as  $a_{n+1} = na_n + \frac{n(n-1)}{2}$ . Divide by  $n!$  on both sides to obtain:

$$\frac{a_{n+1}}{n!} = \frac{a_n}{(n-1)!} + \frac{1}{2(n-2)!}.$$

Let  $c_{n+1} = a_{n+1}/n!$  for  $n \geq 0$ . Then we may rewrite the above recurrence as

$$c_{n+1} = c_n + \frac{1}{2(n-2)!},$$

where  $n \geq 2$ . Note that  $c_2 = a_2/1! = a_2 = 0$ . This gives us  $c_{n+1} = \sum_{p=0}^{n-2} \frac{1}{2p!}$  for  $n \geq 2$ . Since  $a_{n+1} = n!c_{n+1}$ , we have

$$a_{n+1} = n! \sum_{p=0}^{n-2} \frac{1}{2p!},$$

for  $n \geq 2$ . The initial conditions are  $a_0 = a_1 = a_2 = 0$ .

### Problem 6-2

Suppose that the following program is called from a main function and let  $a_n$  be the number of calls to  $f(n, k)$ .

1. Determine the recurrence of  $a_n$  and express it in terms of the shift operator  $\mathbf{E}$ .
2. Reduce the degree of the recurrence by factorizing the operator expression. Solve the newly obtained recurrence  $b_n$ .

```

int f(n, k) {
    int sum = k;
    if(n > 1) {
        for(int i = 0; i <= n; ++i){
            sum += f(n - 1, i);
            for(int j = 0; j < n; ++j){
                sum += f(n - 2, i + j);
                sum += f(n - 2, i + 2 * j);
            }
        }
    } else {
        sum = n;
    }
    return sum;
}

```

### Solution

First note that the second parameter in the function does not influence the number of recursive calls. For  $n \in \{0, 1\}$ , the function  $f$  is called once and hence  $a_0 = a_1 = 1$ . For  $n \geq 2$ , there is one call to  $f(n)$  and inside the function there are  $n + 1$  calls to  $f(n - 1)$  and  $2n(n + 1)$  calls to  $f(n - 2)$ . Therefore  $a_n$  may be written as

$$a_n = (n + 1)a_{n-1} + 2n(n + 1)a_{n-2} + 1,$$

which in operator notation can be written as

$$(\mathbf{E}^2 - (n + 1)\mathbf{E} - 2n(n + 1))a_{n-2} = 1.$$

The operator  $\mathbf{E}^2 - (n + 1)\mathbf{E} - 2n(n + 1)$  can be factorized as  $(\mathbf{E} - 2(n + 1))(\mathbf{E} + n)$ . We let  $(\mathbf{E} + n)a_{n-2} = b_{n-2}$ , where  $n \geq 2$ . Note that  $b_0 = a_1 + 2 \cdot a_0 = 3$ . Our first reduced recurrence is therefore:  $(\mathbf{E} - 2(n + 1))b_{n-2} = 1$ , which may be rewritten as:

$$b_n - 2(n + 2)b_{n-1} = 1 \text{ where } n \geq 1 \text{ and } b_0 = 3.$$

Dividing both sides by  $2^n \cdot (n + 2)!$ , we obtain:

$$\frac{b_n}{2^n \cdot (n + 2)!} = \frac{b_{n-1}}{2^{n-1} \cdot (n + 1)!} + \frac{1}{2^n \cdot (n + 2)!}.$$

Setting  $c_n = \frac{b_n}{2^n \cdot (n + 2)!}$ , where  $n \geq 0$ , we obtain that:

$$c_n = \frac{b_0}{2!} + \sum_{j=1}^n \frac{1}{2^j \cdot (j + 2)!} = \frac{3}{2} + \sum_{j=1}^n \frac{1}{2^j \cdot (j + 2)!}.$$

Therefore  $b_n$  is given by:

$$b_n = 2^n \cdot (n + 2)! \cdot \left( \frac{3}{2} + \sum_{j=1}^n \frac{1}{2^j \cdot (j + 2)!} \right).$$

**Homework Assignment 6-1 (10 Points)**

Provide a closed form for  $a_n$  from Problem 6-2 (up to initial conditions), using the solution obtained for  $b_n$  in class.

**Solution** See above.

**Homework Assignment 6-2 (10 Points)**

Solve the following recurrence by reducing its degree.

$$\begin{aligned} a_0 &= 8000 \\ a_1 &= \frac{1}{2} \\ a_{n+2} + a_{n+1} - n^2 a_n &= n! \end{aligned}$$

**Solution**

$$\begin{aligned} a_{n+2} + a_{n+1} - n^2 a_n &= (\mathbf{E} - n) \underbrace{(\mathbf{E} + n)a_n}_{b_n} \\ (\mathbf{E} - n)b_n &= n! \\ b_{n+1} - nb_n &= n! \\ b_n &= n! \\ (\mathbf{E} + n)a_n &= n! \\ a_{n+1} + na_n &= n! \\ a_n &= \frac{(n-1)!}{2} \end{aligned}$$

for  $n > 0$  and  $a_0 = 8000$ .