Analysis of Algorithms

Problem 5-1

Solve the following recurrence: $na_n = (n+1)a_{n-1} + 2n$ for $n \geq 1$ and $a_0 = 0$.

Solution:

We may write the recurrence as: $a_n = (1+\frac{1}{n})a_{n-1} + 2$ and using the "summation factor" technique write down the solution as:

$$\begin{aligned} a_n &= 2 + \sum_{j=1}^{n-1} 2\left(1+\frac{1}{j+1}\right)\cdot\left(1+\frac{1}{j+2}\right)\cdot\left(1+\frac{1}{n}\right) \\ &= 2 + \sum_{j=1}^{n-1} 2\left(\frac{j+2}{j+1}\right)\cdot\left(\frac{j+3}{j+2}\right)\cdot\left(\frac{n+1}{n}\right) \\ &= 2 + \sum_{j=1}^{n-1} 2\frac{n+1}{j+1} \\ &= 2 + 2(n+1)\left(\frac{1}{2}+\frac{1}{3}+\cdots+\frac{1}{n}\right) \\ &= 2 + 2(n+1)(H_n - H_1). \end{aligned}$$

Problem 5-2

Compute the number of iterations of the `while`-loop for $0 < i$ and arbitrary j.

```
while ( i <= j ) {
   i = i + j;
   if ( i > j ) then j = j + 10;
}
```

Solution:

Let i_n and j_n be the numbers i, j after the nth iteration of the while-loop. For $i_0 > j_0$ there are no iterations. Let therefore $0 < i_0 \leq j_0$ in the following. We get the recurrence

$$\begin{aligned} i_n &= i_{n-1} + j_{n-1} \\ j_n &= j_{n-1} + 10. \end{aligned}$$

This immediately yields

$$\begin{aligned}
j_n &= j_0 + 10n \\
i_n &= i_{n-1} + 10(n-1) + j_0 \\
&= i_0 + \sum_{k=1}^{n} (10(k-1) + j_0) \\
&= i_0 + 5n(n-1) + nj_0.
\end{aligned}$$

The `while`-loop is iterated while $i_n - j_n \leq 0$ holds, i.e., as long as

$$5n^2 + (j_0 - 15)n + i_0 - j_0 \leq 0.$$

For positive n this inequality holds while

$$n \leq \frac{15 - j_0 + \sqrt{(j_0 - 15)^2 - 20(i_0 - j_0)}}{10} =: a(i_0, j_0).$$

Therefore, the `while`-loop is iterated $\lfloor a(i_0, j_0) \rfloor + 1$ times.

Problem 5-2

Given an array a of length n, an algorithm compares all pairs $(a[i], a[j])$ for all $i < j \leq n$, and then calls itself recursively on all proper prefixes of a.

How often does the algorithm compare two pairs? Use the repertoire method!

Solution: The recurrence is $R_0 = 0$ and

$$R_n = \binom{n}{2} + \sum_{k=0}^{n-1} R_k$$

for $n \geq 1$. Testing a couple of values for R_n, we obtain the repertoire:

R_0	R_n	$f_n = R_n - \sum_{k=0}^{n-1} R_k$
1	a^n	$\frac{2-a}{1-a} a^n - \frac{1}{1-a}$
1	2^n	1
1	1	$1 - n$
0	n	$n - \binom{n}{2}$

To get $\binom{n}{2}$, we can use the last line. To get rid of the linear and constant factors, we use the third and finally the second line, and obtain

$$-\left(n - \binom{n}{2}\right) - (1 - n) + 1 = \binom{n}{2}.$$

Fortunately, this also holds for $R_0 = 0$, and the solution is

$$R_n = 2^n - n - 1.$$

Homework Assignment 5-1 (10 Points)

Use summation factors to solve the following recurrence:

$$\begin{aligned} a_0 &= 0 \\ a_n &= \frac{a_{n-1}}{n} + \frac{1}{(n-1)!} \qquad \text{for } n \geq 1 \end{aligned}$$

Solution: Plugging $y_n = 1/(n-1)!$ and $x_n = 1/n$ into the formula known from the lecture yields:

$$\begin{aligned} a_n &= \frac{1}{(n-1)!} + \sum_{j=1}^{n-1} \frac{1}{(j-1)!} \frac{1}{j+1} \cdots \frac{1}{n} \\ &= \frac{1}{(n-1)!} + \sum_{j=1}^{n-1} \frac{j}{n!} \\ &= \frac{1}{(n-1)!} + \frac{1}{n!} \frac{n(n-1)}{2} \\ &= \frac{2n}{2n!} + \frac{n(n-1)}{2n!} \\ &= \frac{n+1}{2(n-1)!} \end{aligned}$$

Homework Assignment 5-2 (10 Points)

Use the repertoire method to find a closed form for the following recurrence:

$$\begin{aligned} a_0 &= 5 \\ a_1 &= 9 \\ a_n &= na_{n-1} + n^2 a_{n-2} - n^4 - 3n^2 + 5 \qquad \text{for } n \geq 2 \end{aligned}$$

Solution: Let $f(n) = -n^4 - 3n^2 + 5$, i.e., $f(n) = a_n - na_{n-1} - n^2 a_{n-2}$.

a_n	$f(n)$	a_0	a_1
1	$-n^2 - n + 1$	1	1
n	$-n^3 + n^2 + 2n$	0	1
n^2	$-n^4 + 3n^3 - n^2 - n$	0	1

Let Z_i for $i = 1, 2, 3$ be the solutions of the first, second, and third line, respectively. Then $f(n) = 5Z_1 + 3Z_2 + Z_3$. For these, a_0 and a_1 are correct, and thus $a_n = 5 \cdot 1 + 3n + n^2$.