Analysis of Algorithms

Exercise 4-1
Solve the following recurrence: Let $a_0 = 1$, $a_1 = 1$, $a_2 = 4$ and

$$a_n = 2a_{n-1} - a_{n-2} + 2a_{n-3}, \text{ for } n \geq 3.$$

Solution:
The characteristic polynomial is $z^3 - 2z^2 + z - 2$. This can be factorized as $(z^2+1)(z-2)$, which gives the set of roots as $\{\pm i, 2\}$. The general solution is thus:

$$a_n = \alpha \cdot 2^n + \beta \cdot (-i)^n + \gamma \cdot i^n.$$

Substituting the values for $n = 0, 1, 2$, we obtain the following system of linear equations:

$$\alpha + \beta + \gamma = 1 \tag{1}$$
$$2\alpha - i\beta + i\gamma = 1 \tag{2}$$
$$4\alpha - \beta - \gamma = 4 \tag{3}$$

Solving this system yields $\alpha = 1$, $\beta = -i/2$ and $\gamma = i/2$. Note that β and γ are complex conjugates. The solution is thus:

$$a_n = 2^n - \frac{i}{2} \cdot (-i)^n + \frac{i}{2} i^n \tag{4}$$
$$= 2^n + (1 + (-1)^{n+1}) \cdot \frac{i^{n+1}}{2}. \tag{5}$$

This solution can be written in a much nicer form using Euler's formula:

$$e^{i\theta} = \cos\theta + i \sin\theta.$$

We may write $i^{n+1}/2$ as follows:

$$\frac{i^{n+1}}{2} = \frac{1}{2}\left(\cos\frac{\pi}{2} + i \sin\frac{\pi}{2}\right)^{n+1}$$
$$= \frac{e^{\frac{i\pi}{2}\cdot(n+1)}}{2}.$$

Similarly, $(-1)^{n+1} i^{n+1}/2$ may be written as $\frac{1}{2} \cdot e^{-\frac{i\pi}{2}\cdot(n+1)}$. Now note that

$$\frac{e^{\frac{i\pi}{2}\cdot(n+1)} + e^{-\frac{i\pi}{2}\cdot(n+1)}}{2} = \cos\left(\frac{\pi(n+1)}{2}\right).$$

Thus $a_n = 2^n + \cos\frac{\pi(n+1)}{2}$.

Exercise 4-2

Solve the following recurrence: Let $b_1 = b_2 = b_3 = 1$ and

$$b_n = 3b_{n-1} - 4b_{n-2} + 12b_{n-3} \text{ for } n > 3.$$

Solution:

Here the characteristic polynomial is $z^3 - 3z^2 + 4z - 12$. This may be factorized as:

$$\begin{aligned} z^3 - 3z^2 + 4z - 12 &= z^2(z-3) + 4(z-3) \\ &= (z^2+4)(z-3) \\ &= (z+2i)(z-2i)(z-3). \end{aligned}$$

Since the roots are $\{\pm 2i, 3\}$, the general solution is of the form

$$b_n = \lambda \cdot 3^n + \mu \cdot (2i)^n + \delta \cdot (-2i)^n.$$

Using the known values for $n = 1, 2, 3$, we obtain a system of equations with three variables λ, μ, δ that can be solved with the standard methods. We actually have $\lambda = \frac{20}{3 \cdot 52}$, $\mu = \frac{1-8i}{52}$ and $\delta = \frac{1+8i}{52}$.

Homework Assignment 4-1 (10 Points)

Solve the following recurrence: Let $a_0 = 0$, $a_1 = 3$ and

$$a_n = 4a_{n-1} - 4a_{n-2} \text{ for } n > 1.$$

Solution: The characteristic polynomial is $x^2 - 4x + 4 = (x-2)^2$ with the only root $x_0 = 2$. The solution is therefore of form $a_n = \lambda 2^n + \mu n 2^n$. Our constraints yield $\lambda = 0$ and $\mu = \frac{3}{2}$.

Homework Assignment 4-2 (10 Points)

Solve the following recurrence and find a nice representation of the solution (in a mathematical sense).

$$\begin{aligned} c_0 &= 2 \\ c_1 &= 4 \\ c_n &= c_{n-2}^{\log c_{n-1}} \end{aligned}$$

Solution: We apply the logarithm and obtain

$$\log c_n = \log c_{n-1} \cdot \log c_{n-2}.$$

In order to obtain a sum instead of a product, we repeat this and obtain

$$\log \log c_n = \log \log c_{n-1} + \log \log c_{n-2}.$$

Substituting $d_n = \log \log c_n$ yields

$$\begin{aligned} d_0 &= 0 \\ d_1 &= 1 \\ d_n &= d_{n-1} + d_{n-2} \end{aligned}$$

Since this describes the Fibonacci numbers, we obtain $c_n = 2^{2^{F_n}}$, where F_n denotes the n-th Fibonacci number.