RWTH Aachen **SS 2012**
Lehrgebiet Theoretische Informatik Tutorial 11
Langer—Reidl—Rossmanith—Sikdar 2.7.2012

Analysis of Algorithms — Tutorial

Exercise 11-1

Sort the following generating functions *within one minute* by their exponential growth!

1. $A(z) = 1/\sqrt{1 - z/2}$
2. $B(z) = (1 + z)/(1 - z)$
3. $C(z) = \frac{1}{1-e^{-z-1/3}}$

Solution: We only need to sort them by the absolute value of the dominant singularities. $A_n \asymp 1/2^n$, $B_n \asymp 1$, $C_n \asymp 3^n$.

Exercise 11-2

An algorithm is given an array of length $n \geq 0$ and, if $n \geq 2$, for each $1 \leq k \leq n$ calls itself on some random subarray of length k with probability $\frac{1}{2}$. Compute the exponential growth of the running time of this algorithm.

Solution:

Let t_n be the expected number of calls of the algorithm for an array of length n. Then $t_0 = t_1 = 1$ and $t_n = 1 + \frac{1}{2} \sum_{k=1}^{n} t_k$ for $n > 1$. To get an equation that holds for all n, we let

$$t_n = 1 + \frac{1}{2} \sum_{k=1}^{n} t_k - \frac{1}{2}(n = 1).$$

The corresponding OGF $T(z)$ is

$$T(z) = \frac{1}{1-z} + \frac{1}{2}\frac{1}{1-z}T(z) - \frac{1}{2}z,$$

where we used the convolution

$$\frac{1}{1-z}T(z) = \sum_{n\geq 0} 1z^n \sum_{n\geq 0} t_n z^n = \sum_{n\geq 0}\sum_{k=0}^{n} 1 t_k z^n = \sum_{n\geq 0}\sum_{k=1}^{n} t_k z^n = \sum_{n\geq 0} t_n z^n.$$

Solving the equation for $T(z)$ leads to $T(z) = \frac{z^2-z+2}{1-2z}$. Since the dominant singularity is located at $z = \frac{1}{2}$, we get an asympotic running time of $\asymp 2^n$.

Exercise 11-3
Determine $[z^n]\frac{1}{2-e^z}$ up to an additive error of $O(12^{-n})$!
Solution: We first determine the dominant singularity, which is $\ln 2$. Since, by L'Hôpital,

$$\lim_{z\to\ln 2}\frac{\ln 2 - z}{2-e^z}=\frac{1}{2},$$

we have

$$\frac{\ln 2 - z}{\ln 2 - z}\cdot\frac{1}{2-e^z}\sim\frac{1}{2}\cdot\frac{1}{\ln 2 - z}=\frac{1}{2\ln 2}\cdot\frac{1}{1-(1/\ln 2)z}=\frac{1}{2\ln 2}\sum_{n=0}^{\infty}\left(\frac{1}{\ln 2}\right)^n z^n.$$

We now look for the next singularites (ordered by their distance from the origin). These are $\ln 2 \pm 2\pi i$. By L'Hôpital

$$\lim_{z\to\ln 2\pm 2\pi i}\frac{z-\ln 2\mp 2\pi i}{2-e^z}=-\frac{1}{2}$$

und therefore

$$S(z)\sim-\frac{1}{2}\frac{1}{z-\ln 2\mp 2\pi i}=\frac{1}{2}\frac{1}{\ln 2\mp 2\pi i}\frac{1}{1-\frac{z}{\ln 2\mp 2\pi i}}$$

for $z\to\ln 2\pm 2\pi i$. We can now use the Theorem of the lecture to determine the coefficients:

$$\begin{aligned}
[z^n]S(z) &= \frac{1}{2}\left((\frac{1}{\ln 2})^{n+1}+(\frac{1}{\ln 2+2\pi i})^{n+1}+(\frac{1}{\ln 2-2\pi i})^{n+1}\right)+O(r)^{-n}\\
&= \frac{1}{2}\left((\frac{1}{\ln 2})^{n+1}+r^{n+1}(e^{i\phi(n+1)}+e^{-i\phi(n+1)})\right)+O(r)^{-n}\\
&= \frac{1}{2}\left((\frac{1}{\ln 2})^{n+1}+r^{n+1}\cos(\phi(n+1))\right)+O(r)^{-n}
\end{aligned}$$

with $r=1/\sqrt{\ln^2 2+4\pi^2}\approx 12.58547409739904$, $\phi=\arctan(\frac{2\pi}{\ln 2})$.

Homework Assignment 11-1 (10 points)
Determine the exponential growth of $[z^n]G(z)$, where

a) $G(z)=z^2/(1-z-z^2)$,

b) $G(z)=\sqrt{1+2z}-\sqrt{2+2z-4z^2}/\sqrt{3}$,

c) $G(z)=\ln(1+\sin(z))/\ln(1+\cos(z))$.

Solution:

a) We solve $1-z-z^2=0$, which yields $z=-\frac{1}{2}\pm\frac{1}{2}\sqrt{5}$. The dominant singularity is at $\phi=\frac{1}{2}\sqrt{5}-\frac{1}{2}\approx 0.6181$. Hence, $[z^n]G(z)\asymp\phi^{-n}\approx 1.6181^n$.

b) The dominant singularity is at $-\frac{1}{2}$. Therefore, $[z^n]G(z)\asymp 2^n$.

c) Using the rules for the logarithm, we get $G(z)=\ln(\sin(z)-\cos(z))$. The dominant singularity is at $\pi/4$. Therefore, $[z^n]G(z)\asymp(4/\pi)^n\approx 1.2732^n$.

Homework Assignment 11-2 (10 points)

$$A(z) = \frac{\sqrt{1-z^7}}{2z^2-3z+1} \qquad B(z) = \frac{1-z^2}{e^{z+3z^2}} \qquad C(z) = z^5 + 3z^2(z^3+z^2+8)$$

Sort the sequences a_n, b_n, and c_n by their exponential growth.

Solution:

We have $c_n = 0$ for almost all n. The sequence b_n growth subexponentially, because $B(z)$ lacks a singularity. The dominant singularity of $A(z)$ is at $\frac{1}{2}$, i.e., $[z^n]A(z) \asymp 2^n$.

Homework Assignment 11-3 (10 Points)

Determine $g_n = [z^n]G(z)$ up to an additive error of $O(4^n)$, where

$$G(z) = \sum_{n=0}^{\infty} g_n z^n = \frac{15z^2+8z+1}{15z^2-8z+1}.$$

Solution: We have

$$G(z) = \frac{15z^2+8z+1}{(3z-1)(5z-1)}.$$

Its singularities are at $\frac{1}{3}$ und $\frac{1}{5}$, both being poles of order 1. Since

$$G(z) \sim \frac{8}{1-5z} \text{ für } z \to \frac{1}{5},$$

we have $g_n - [z^n]\frac{8}{1-5z} = O((3+\epsilon)^n)$ for any $\epsilon > 0$. Therefore, $g_n = 8 \cdot 5^n + O(4^n)$.