RWTH Aachen **SS 2012**
Lehrgebiet Theoretische Informatik Tutorial 10
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Analysis of Algorithms — Tutorial

Problem 10-1

Find the generating function for the series given by the following recurrence relation:

$$f_n = f_{n-1} + 2f_{n-2} + 3f_{n-3} + \cdots + nf_0 \text{ for } n > 0 \text{ and } f_0 = 1.$$

Solution: We first need to find a formula for f_n that holds for all n (with $f_n = 0$ für $n < 0$). For, we add 1 if $n = 0$. The equation is

$$f_n = (n = 0) + \sum_{i=1}^{n} i f_{n-i}.$$

We now switch to the power series, which yields

$$\sum_{n=0}^{\infty} f_n z^n = \sum_{n=0}^{\infty} ((n = 0) + \sum_{i=1}^{n} i f_{n-i}) z^n = 1 + \sum_{n=0}^{\infty} \sum_{i=0}^{n} i f_{n-i} z^n.$$

The right term is the convolution, and we rewrite the equation as

$$F(z) = \frac{zF(z)}{(1-z)^2} + 1,$$

where $1/(1-z)^2$ is the GF for the sequence $a_n = n + 1$. Solving for $F(z)$ yields

$$F(z) = \frac{(1-z)^2}{z^2 - 3z + 1}.$$

Problem 10-2

A software engineer is writing a program that needs at some point a random permutation without fixpoints of the numbers $1, \ldots, n$. A permutation π has a fixpoint k if $\pi(k) = k$. To be more precise: The program has to choose one permutation among all permutations without fixpoints with equal probability.

She wants to reuse as much software as possible and finds a library routine that provides random permutations. Her plan is to call that routine until it delivers a fixpoint free permutation. She worries a bit, however, how many calls she has to make on average.

a) If $\pi \in S_n$ is a random permutation, what is the expected number of fixpoints? Why is the answer not very helpful to her task?

b) Let q_n be the number of fixpoint-free permutations in S_n. Find the EGF for q_n. Start with a recurrence and use the algebraic rules for EGFs.

c) Find the first values of q_n with the help of a computer algebra system.

Solution:

a) For $1 \le i \le n$ we let X_i be binary variable that i is a fixpoint in a random permutation over n elements. Then $E(X_i) = 1/n$, because among the $n!$ permutations there are $(n-1)!$ permutations with a fixpoint at i. The total number of fixpoints in a random permutation is then $\sum_{i=1}^n X_i$, and, using the linearity of expected values, the expected number of fixpoints in a random permutation is

$$E\big(\sum_{i=1}^n X_i\big) = \sum_{i=1}^n E(X_i) = \sum_{i=1}^n 1/n = 1.$$

b) Let D_m be the number of *derangements* over a set of order m. In order to find an equation that contains D_m, we try to express $P_{n,k}$, denoting the number of permutations over a set of order n that has exactly k fixpoints, with the help of D_m.

Firstly, note that there are $\binom{n}{k}$ possibilities to distribute the k fixpoints over n. To get *exactly* k fixpoitns, we require the $n-k$ remaining elements to be no fixpoints, i.e., they form a derangement over the remaining elements. Therefore,

$$P_{n,k} = \binom{n}{k} D_{n-k}.$$

We now take the sum over all $k \in [0, n]$, because this yields *all* permutations, whose number ($n!$) we know. This yields

$$\sum_{k=0}^n P_{n,k} = n! = \sum_{k=0}^n \binom{n}{k} D_{n-k}.$$

We now switch to EGFs:

$$\frac{1}{1-z} = D(z)e^z$$

Then

$$D(z) = \frac{e^{-z}}{1-z}.$$

c)
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Maxima 5.23.2 http://maxima.sourceforge.net
using Lisp SBCL 1.0.38-3.el6
Distributed under the GNU Public License. See the file COPYING.
Dedicated to the memory of William Schelter.
The function bug_report() provides bug reporting information.
(%i1) t(z) := exp(-z)/(1-z);
                                             exp(- z)
(%o1)                             t(z) := --------
                                              1 - z
(%i2) taylor(t(z), z, 0, 5);
                           2     3      4       5
                          z     z    3 z    11 z
(%o2)/T/              1 + -- + -- + ---- + ----- + . . .
                          2     3     8      30
```

Homework Assignment 10-1 (10 points)
Expand the generating function from Problem 10-1 in order to find a closed formula for f_n.
Solution
We have

$$F(z) = \frac{(1-z)^2}{(z^2-3z+1)} = 1 + \frac{z}{z^2-3z+1}.$$

One may factor $z^2 - 3z + 1$ into $(z - \frac{3-\sqrt{5}}{2})(z - \frac{3+\sqrt{5}}{2})$. Using the technique of partial fractions (which we do not describe here), one can write $F(z)$ as follows:

$$F(z) = 1 + \frac{1}{\sqrt{5}}\left[\frac{1}{1-\frac{2}{3-\sqrt{5}}z} - \frac{1}{1-\frac{2}{3+\sqrt{5}}z}\right].$$

we may now use the series expansion $1/(1-az) = \sum_{n=0}^{\infty}(az)^n$ to write $F(z)$ as

$$\begin{aligned}
F(z) &= 1 + \frac{1}{\sqrt{5}}\left[\sum_{n=0}^{\infty}\left(\frac{2}{3-\sqrt{5}}z\right)^n - \sum_{n=0}^{\infty}\left(\frac{2}{3+\sqrt{5}}z\right)^n\right] \\
&= 1 + \frac{1}{\sqrt{5}}\sum_{n=0}^{\infty}\left[\left(\frac{2}{3-\sqrt{5}}\right)^n - \left(\frac{2}{3+\sqrt{5}}\right)^n\right]z^n \\
&= 1 + \frac{1}{\sqrt{5}}\sum_{n=0}^{\infty}\left[\left(\frac{3+\sqrt{5}}{2}\right)^n - \left(\frac{3-\sqrt{5}}{2}\right)^n\right]z^n.
\end{aligned}$$

Therefore,

$$f_n = (n=0) + \frac{1}{\sqrt{5}}\left[\left(\frac{3+\sqrt{5}}{2}\right)^n - \left(\frac{3-\sqrt{5}}{2}\right)^n\right].$$

Homework Assignment 10-2 (10 Points)
We call a sequence of **push** und **pop** operations ($\uparrow$ and $\downarrow$) *valid*, if it contains the same number of $\uparrow$ and $\downarrow$ and no prefix of the sequence consists of less $\uparrow$ than $\downarrow$. For example, $(\uparrow,\uparrow,\downarrow,\downarrow,\uparrow,\downarrow)$ is valid, while $(\downarrow,\downarrow,\uparrow,\uparrow)$ and $(\uparrow,\downarrow,\downarrow,\uparrow)$ are not valid. The number of $\uparrow$s in a valid sequence is called the *length* of the sequence.
How many valid sequences of length n do exist?

Solution
Let p_n be the number of valid sequences of length n. Using the symbolic method, we have

$$P = (\uparrow P \downarrow)^*.$$

The generating function is therefore

$$P(z) = \frac{1}{1-zP(z)}.$$

We immediately get $1 - P(z) + zP(z)^2 = 0$, which yields two solutions. One of those is not of interest for our purposes, the other one is:

$$
\begin{aligned}
P(z) &= -\frac{\sqrt{1-4z}-1}{2z} \\
&= -\frac{1}{2z}(\sum_{n\geq 0}\binom{\frac{1}{2}}{n}(-4)^n z^n - 1) \\
&= -\frac{1}{2}(\sum_{n\geq 0}\binom{\frac{1}{2}}{n}(-4)^n z^{n-1} - \frac{1}{z}) \\
&= -\frac{1}{2}(\sum_{n\geq -1}\binom{\frac{1}{2}}{n+1}(-4)^{n+1} z^{n} - \frac{1}{z})
\end{aligned}
$$

Therefore:

$$
\begin{aligned}
p_n = [z^n]P(z) &= -\frac{1}{2}\binom{\frac{1}{2}}{n+1}(-4)^{n+1} \\
&= -\frac{1}{2}(-1)^n\frac{1}{4^{n+1}}\frac{1}{2n+1}\binom{2n+2}{n+1} \\
&= \frac{1}{4n+2}\binom{2n+2}{n+1}
\end{aligned}
$$