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SS 2012
Problem Set 6
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Analysis of Algorithms — Tutorial

Problem 6-1

Use the operator technique to obtain a solution to the following recurrence:

$$a_{n+2} = (n+2)a_{n+1} - na_n + n,$$

with $a_0 = 0 = a_1$.

Solution

We may rewrite the recurrence in operator notation as follows:

$$(E^2 - (n+2)E + n)a_n = n.$$

The polynomial $E^2 - (n+2)E + n$ can be factored into $(E-1)(E-n)$, using the indentity $En = (n+1)E$. We now let $b_n = (E-n)a_n$, for $n \geq 0$, and solve for b_n. We have $(E-1)b_n = n$, which written in full, amounts to $b_{n+1} = b_n + n$, where $n \geq 0$ and $b_0 = a_1 = 0$. This gives

$$b_n = b_0 + \frac{n(n-1)}{2} = \frac{n(n-1)}{2}.$$

We now solve for a_n. We have $(E-n)a_n = b_n$ and this may be written out as $a_{n+1} = na_n + \frac{n(n-1)}{2}$. Divide by $n!$ on both sides to obtain:

$$\frac{a_{n+1}}{n!} = \frac{a_n}{(n-1)!} + \frac{1}{2(n-2)!}.$$

Let $c_{n+1} = a_{n+1}/n!$ for $n \geq 0$. Then we may rewrite the above recurrence as

$$c_{n+1} = c_n + \frac{1}{2(n-2)!},$$

where $n \geq 2$. Note that $c_2 = a_2/1! = a_2 = 0$. This gives us $c_{n+1} = \sum_{p=0}^{n-2} \frac{1}{2p!}$ for $n \geq 2$. Since $a_{n+1} = n!c_{n+1}$, we have

$$a_{n+1} = n! \sum_{p=0}^{n-2} \frac{1}{2p!},$$

for $n \geq 2$. The initial conditions are $a_0 = a_1 = a_2 = 0$.

Problem 6-2
Let a_n be the number of calls to $f(n, k)$ in the following program.

```
int f(n, k) {
    int sum = k;
    if(n > 1){
        for(int i = 0; i ≤ n; i++){
            sum += f(n − 1, i);
            sum += f(n − 1, 2 ∗ i);
            sum += f(n − 1, 3 ∗ i);
            for(int j = 0; j < n; j++){
                sum += f(n − 2, i + j);
                sum += f(n − 2, i + 2 ∗ j);
            }
        }
    } else {
        sum = n;
    }
    return sum;
}
```

1. Determine the recurrence of a_n and express it in terms of the shift operator E.
2. Reduce the degree of the recurrence by factorizing the operator expression. Solve the newly obtained recurrence b_n.

Solution For $n = 1$, the function f is called once and hence $a_1 = 1$. Define $a_0 = 0$. Now a_n may be written as

$$a_n = 3(n+1)a_{n-1} + 2n(n+1)a_{n-2},$$

which in operator notation can be written as $(E^2 - 3(n+1)E - 2n(n+1))a_{n-2} = 0$. The operator $(E^2 - 3(n+1)E - 2n(n+1))$ can be factored into $(E - 2(n+1))(E - n)$. We let $(E - n)a_{n-2} = b_{n-2}$, where $n \geq 2$. Note that $b_0 = a_1 - 2 \cdot a_0 = 1$. This gives us $b_{n-2} = 2^{n-3}n!$. Now we solve for a_{n-2} using the fact that $(E - n)a_{n-2} = b_{n-2} = 2^{n-3}n!$. This gives us $a_{n-1} - na_{n-2} = 2^{n-3}n!$. Dividing by $n!$ throughout, we obtain

$$\frac{a_{n-1}}{n!} = \frac{a_{n-2}}{(n-1)!} + 2^{n-3}.$$

We let $c_{n-1} = a_{n-1}/n!$ for $n \geq 2$. Then $c_1 = a_1/2! = 1/2$ and from the recurrence we obtain that $c_1 = c_0 + 1/2$, and hence we must define $c_0 = 0$. Also note that $c_2 = c_1 + 2^0$. By expanding the recurrence we obtain $c_{n-1} = 2^{n-2} - 1/2$ and since $a_{n-1} = n!c_{n-1}$, we have that $a_{n-1} = n!(2^{n-2} - 1/2)$. The final solution is

$$a_n = (n+1)!\left(2^{n-1} - \frac{1}{2}\right),$$

for $n \geq 2$ with $a_0 = 0$ and $a_1 = 1$.

Homework Assignment 6-1 (10 Points)
Provide a closed form for a_n from Problem 6-2 (up to initial conditions), using the solution obtained for b_n in class[1].

Solution See above.

Homework Assignment 6-2 (10 Points)
Solve the following recurrence by reducing its degree.

$$\begin{aligned} a_0 &= 8000 \\ a_1 &= \frac{1}{2} \\ a_{n+2} + a_{n+1} - n^2 a_n &= n! \end{aligned}$$

Solution

$$\begin{aligned} a_{n+2} + a_{n+1} - n^2 a_n &= (E-n)\underbrace{(E+n)a_n}_{b_n} \\ (E-n)b_n &= n! \\ b_{n+1} - nb_n &= n! \\ b_n &= n! \\ (E+n)a_n &= n! \\ a_{n+1} + na_n &= n! \\ a_n &= \frac{(n-1)!}{2} \end{aligned}$$

for $n > 0$ and $a_0 = 8000$.

[1] $b_{n-2} = 2^{n-3} n!$